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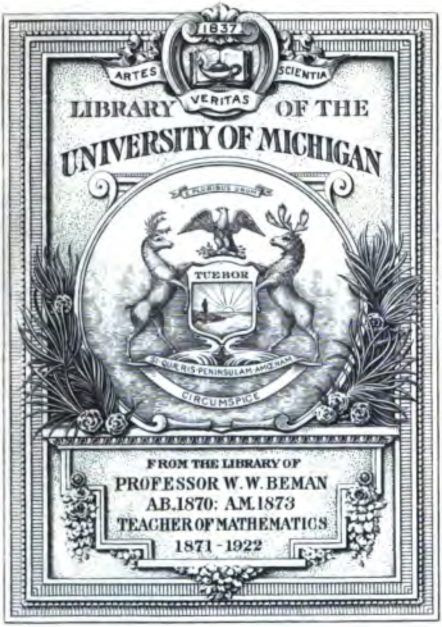

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TREATISE

ON

THE CALCULUS

OF

VARIATIONS.

ILLUSTRATED BY NUMEROUS EXAMPLES.

SECOND EDITION.

E. Dwight Townsend

THE ELEMENTS

OF

PLANE AND SPHERICAL TRIGONOMETRY;

AND ITS APPLICATION TO

ASTRONOMY, DIALLING, AND

TRIGONOMETRICAL SURVEYING.

WITH PLATES.

DESIGNED FOR MATHEMATICAL STUDENTS.

BY RICHARD ABBATT, F.R.A.S., &c.

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PREFACE.

IN presenting these ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY to the notice of the Public, it has been the author's aim to embrace as much of the theory and practice as appeared to be consistent with the time usually devoted to this branch of analysis in a system of liberal education; and to exhibit the subject agreeably to that systematic reasoning, and those generalizations, which constitute the chief beauty and excellence of mathematical learning.

The first section contains the usual principles and definitions. In defining the trigonometrical lines, cosine, cotangent, and cosecant*, they have been re-

---

* When these terms were first introduced, the Science of Trigonometry was confined to a single object, as may be seen by the words τριγωνος, and μετρεω; and therefore it was seldom or never necessary to consider these terms as applied otherwise than to the solution of triangles, or to arcs and angles less than 180°, in which little difficulty

425698

garded as independent of the complement, principally for the sake of uniformity, and for the purpose of affording greater facility to the learner in acquiring a knowledge of their positions throughout the four quadrants.

In the second section, the principal theorems commonly called the Arithmetic of Sines, have been given. They have been arranged in such a manner, that the student, it is presumed, will experience no difficulty in turning immediately to any formula he may want, when the work is used as a reference in the further continuance of his studies.

The third section explains the nature of subsidiary angles, and illustrates their use in computing the numerical values of formulæ not immediately adapted to logarithmic calculation.

In the solution of plane and spherical triangles

---

could arise respecting the sine, &c., of the complement. But in the Arithmetic of Sines, which now forms so prominent a feature in the modern acceptation of Trigonometry, it is not considered sufficient that trigonometrical formulæ should be restricted to arcs of limited magnitude. For the purpose, therefore, of generalizing the demonstrations, as well as for the above reasons, it appeared more simple to define the cosine, cotangent, and cosecant independently of the complement. The usual definitions will then appear in the light of propositions, to which indeed they strictly belong.

contained in the fourth, fifth, and sixth sections, formulæ for every case have been given; those have been pointed out which are generally the most convenient; and care has been exercised that the accompanying examples should enable the student to descend with facility into practical operations.

The seventh section includes such a view of the higher trigonometrical analysis as accorded with the limited nature of the work.

The eighth contains one or two useful articles relating to the circle; and the explanation of the methods of determining the roots of quadratic and cubic equations which are not easily obtained otherwise.

The construction of trigonometrical tables occupies the ninth section, and with this concludes the subject of Trigonometry.

It is always more interesting to young persons studying the principles of any science, to have early opportunities of exercise in the practical application, so important for many obvious reasons. With this view, a variety of examples on the heights and distances of objects have been selected to exemplify Plane Trigonometry: and immediately succeeding the explanation of the Doctrine of the Sphere, a considerable number of Astronomical

Problems have been solved, the accompanying examples to which afford, perhaps, the most obvious exercise;—whilst the concluding three sections, on the Longitude, Dialling, and Geodetic operations, appeared to be in no respect foreign to a work professing to illustrate the use of Spherical Trigonometry.

Such being a short account of the principal subjects in the book, it may be proper, as connected with the general execution, to state the plan which was adopted by the author, previously to its being sent to the press. When the manuscript had been nearly finished, each section was neatly copied, and successively placed in the hands of seven or eight young students, but very slightly acquainted with Trigonometry, whose ages varied from thirteen to eighteen years, with a request to each that he would study the work. By this means, such passages and subjects were discovered, in the understanding of which, *more than the average number* experienced a difficulty. These were consequently examined, and generally either altered or further explained.

It was intended in this manner to compile an elementary work for young persons, which would induce them to look more frequently within themselves for assistance in their mathematical studies than has sometimes been the case, from knowing that it contains few difficulties which they will not be able to overcome. The study of such a work will evidently have a tendency to increase in the learner a confidence in himself, which will grow stronger as he advances, and incite him to continue his mathematical studies, that those principles and methods might be acquired, the influence and use of which extend so widely in the concerns of life. The great object of his early studies, a discipline of mind, will likewise be promoted; the unfailing resource and guide in all his future pursuits.

RICHARD ABBATT.

LONDON, 1841.

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THE
ELEMENTS
OF
PLANE AND SPHERICAL
TRIGONOMETRY.

SECTION I.

Trigonometry is the science which comprehends all theorems respecting the properties of angles and circular arcs, and the lines which it has been found convenient to consider as belonging to them.

DEFINITIONS.

(1.) The circumference of the circle is considered to be divided into 360 equal parts, called *degrees*, each of these is subdivided into 60 equal parts, called *minutes*, each of these into 60, called *seconds*; the second is sometimes divided into 60 *thirds*, but it is more usual to divide it decimally. Degrees, minutes, and seconds, are marked thus $^\circ, ', ''$. The French, however, when they established their declmal system of weights and measures, divided the ~~circumference~~ Semicircumference into 400 equal parts, called *grades*, each grade into 100 equal parts, called *minutes*, and each minute into 100, called *seconds* (*a*)*.

In the following pages we shall generally consider the radius of the circle as the unit of linear measure. The semicircumference will then

$$=3.141592653589793\dagger;$$

$$\text{one degree} = 0.017453292519943;$$

$$\text{one minute} = 0.000290888208666;$$

$$\text{one second} = 0.000004848136811.$$

The value of the semicircumference to radius 1 is generally denoted by π; the value of the quadrant will therefore be $\frac{\pi}{2}$, and that of the circumference 2π.

(2.) Similar arcs of circles are proportional to their radii.

Let C (fig. 1.) be the centre of the two similar arcs AB, ab whose radii are R, r, respectively;

then, Euclid VI. 33,

$$AB : 2\pi R :: \angle C : 360^\circ$$

$$ab : 2\pi r :: \angle C : 360^\circ$$

---

* The small letters within parentheses refer to the notes.

† Vide article (110.)

$\therefore AB : ab :: 2\pi R : 2\pi r :: R : r$;

hence, $\frac{AB}{R} = \frac{ab}{r}$.

(3.) Since the angle is proportional to the arc on which it stands, we may take the arc as a relative value of the angle; but, as it is convenient that the radius should be known and constant, the angle is generally measured by the length of the intercepted arc described with the radius unity; hence, by art. (2.), angle $= \frac{\text{arc}}{\text{radius}}$. On this supposition, the right angle, or $90° = \frac{\pi}{2}$, two right angles, or $180° = \pi$.

(4.) In the higher parts of the science we have frequently to consider arcs as not limited by the circumference, nor even any number of circumferences, and these are still considered to be the measures of angles: in this view, the angle is always measured the same way round the circuit, and therefore it may exceed the limit of the angle contained by any two straight lines, or 180°. As the arc has no limit when conceived to be constantly repeated upon the circumference, so the angle may be of any magnitude whatever, when continually repeated through the circuit.

We proceed now to define the trigonometrical lines belonging to the arc.

(5.) The sine (sin) of an arc is the straight line drawn from the extremity of the arc, perpendicular to the diameter which passes through the origin.

The cosine (cos) is the part of the diameter which passes through the origin intercepted between the centre and the sine.

The tangent (tan) is the straight line touching the arc at the origin, and terminated by the diameter produced, which passes through the end of the arc.

The cotangent (cot) is the straight line which touches the circle at the end of the first quadrant, and terminated by the diameter produced passing through the end of the arc.

The secant (sec) is the straight line drawn from the centre through the end of the arc, and terminated by the tangent.

The cosecant (cosec) is the straight line drawn from the centre through the end of the arc, and terminated by the cotangent.

The versed sine (versin) is the intercepted part of the diameter, between the origin and the sine.

The chord (chd) is the straight line joining the two extremities of the arc.

These definitions are general for all arcs, positive or negative, whatever be their magnitude; but attention must be paid to the signs with which they are affected, according as the arc is terminated in the first, second, third, or fourth quadrant.

(6.) Def. If an arc, or a line measured in one direction, be reckoned *positive,* it must be considered *negative* when similarly measured in the opposite direction.

(7.) Let $ABCD$ (fig. 2.) be a circle, and AC, BD two diameters perpendicular to each other; also suppose AP_1, AP_2, AP_3, AP_4 to be positive arcs, commencing at A, and terminating in the first, second, third and fourth quadrants: draw the trigonometrical lines to each arc according to their definitions in (5.); then, of the arc AP_1, sin, cos, tan, cot, sec, cosec, are respectively, $P_1 M_1$, OM_1, AT_1, Bt_1, OT_1, Ot_1; these lines, belonging to an arc less than the quadrant, it has been agreed to reckon positive.

Of the arc AP_2, sin, cos, tan, cot, sec, cosec, are $P_2 M_2$, OM_2, AT_2, Bt_2, OT_2, Ot_2, respectively;

of which the sine and cosecant are positive, and the others negative.

Of the arc $A P_3$, the above-mentioned quantities are, respectively, $P_3 M_3$, $O M_3$, $A T_3$, $B t_3$, $O T_3$, $O t_3$, of which the tangent and cotangent are positive, and the rest negative*.

Of the arc $A P_4$, they are $P_4 M_4$, $O M_4$, $A T_4$, $B t_4$, $O T_4$, $O t_4$, respectively, where the cosine and secant are positive, and the others negative.

The versed sine and chord are positive for all arcs.

(8.) It appears from the figure that the sine always continues parallel to itself. The tangent commences at the origin of the arc, and the cotangent at the end of the first quadrant; these have always the same position. The cosine, secant, and cosecant commence at the centre of the circle; the position of the cosine is constantly the same, whilst the secant and cosecant vary in position through the circuit.

The following view will show the values of the trigonometrical lines, corresponding to arcs$=0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π, respectively.

---

* The secant and cosecant are negative because they are not measured from the centre *towards* the end of the arc, but *from* it.

Arc.	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	radius	0	—radius	0
cos	radius	0	—radius	0	radius
tan	0	infinite	0	infinite	0
cot	infinite	0	infinite	0	infinite
sec	radius	infinite	—radius	infinite	radius
cosec	infinite	radius	infinite	—radius	infinite

(9.) Since we have considered positive arcs as measured from *A*, in the direction *A B C D*, negative arcs must be measured from the same point, in the direction *A D C B* (6).

Let *A P*, *A P′* (fig. 3.) be two arcs equal in magnitude, the one positive, the other negative; and suppose the trigonometrical lines to be drawn to each arc: then, evidently, $P'M = P M$, $A T' = A T$, $B t' = B t$, $O T' = O T$, $O t' = O t$. Hence, of a negative arc, the cosine and secant are the same as those of an equal positive arc: the sine, tangent, cotangent, and cosecant are equal in respect of magnitude, but affected with contrary signs. In the figure, *A P* was taken less than a quadrant; but it will easily be seen that the same is true, independently of the magnitude of the arc.

(10.) The whole of what has been assumed with respect to the signs to be prefixed to the values of the sines, cosines, &c., is purely arbitrary: its utility consists in lessening the number of formulæ which would otherwise have to be used. A single formula obtained from the consideration of a particular arc, can, by these means, be made general, as we shall see hereafter.

(11.) Def. The defect of an arc from a quadrant is called the *complement* of that arc; and the defect from a semicircle, its *supplement*.

(12.) Let $A\,P$ (fig. 4.) be an arc, and $A\,P'$ its complement, and suppose the trigonometrical lines to be drawn to each; then, by a little consideration, it will appear that $P\,M = O\,M'$, $O\,M = P'\,M'$, $A\,T = B\,t'$, $B\,t = A\,T'$, $O\,T = O\,t'$, $O\,t = O\,T'$. Hence the sin, cos, tan, cot, sec, and cosec of an arc, are, respectively, equal to the cos, sin, cot, tan, cosec, and sec of its complement.

(13.) Make the arc $A\,P_2$ (fig. 3.) = the supplement of $A\,P$, and, in addition to the lines already drawn in the figure, let $P_2\,M_2$ the sine of $A\,P_2$ be drawn; then, by comparing the triangles, it will be seen that the sine and cosecant of an arc are the same as those of the supplement; whilst the cosine, tangent, cotangent, and secant are equal in magnitude, but affected with opposite signs.

SECTION II.

RELATIONS OF TRIGONOMETRICAL LINES.

(14.) Suppose the arc $A\,P_1$ (fig. 2.) to be denoted by A, and let the radius $O\,A = r$. By the similar triangles $O\,M_1\,P_1$, $O\,A\,T_1$,

$$A\,T_1 = \frac{P_1\,M_1\,.\,O\,A}{O\,M_1},\quad O\,T_1, = \frac{O\,P_1\,.\,O\,A}{O\,M_1},$$

or,

$$\tan A = \frac{\sin A\,.\,r}{\cos A},\quad \sec A = \frac{r^2}{\cos A}.$$

(15.) By the similar triangles $P_1\,O\,M_1$, $O\,B\,t_1$,

$$B\,t_1 = \frac{O\,M_1\,.\,O\,B}{P_1\,M_1},\quad O\,t_1 = \frac{O\,P_1\,.\,O\,B}{P_1\,M_1};$$

or,

$$\cot A = \frac{\cos A\,.\,r}{\sin A},\quad \operatorname{cosec} A = \frac{r^2}{\sin A}.$$

Hence, $\tan A\,.\,\cot A = r^2$.

(16.) By Euclid I. 47, $O\,P_1^2 = P_1\,M_1^2 + O\,M_1^2$, $O\,T_1^2 = O\,A^2 + A\,T_1^2$, $O\,t_1^2 = O\,B^2 + B\,t_1^2$,
or,

$$r^2 = \sin{}^2 A + \cos{}^2 A,\ \sec{}^2 A = r^2 + \tan{}^2 A,$$
$$\operatorname{cosec}{}^2 A = r^2 + \cot{}^2 A.$$

(17.) Let A denote the arc $P\,A\,P'$ (fig. 3.) of which the chord is $P\,M\,P'$, and suppose the radius $O\,M\,A$ to bisect both the arc and chord in A and M; then $P\,P' = 2\,P\,M$; hence chd $A = 2 \sin \frac{A}{2}$.

(18.) The theorems in (14.), (15.), and (16.), might seem defective in point of generality, since the arc A, from the consideration of which they were derived is less than a quadrant; but if we pay attention to the conventional signs explained in (7.), we shall find that they hold good for arcs of any magnitude. For example, cos $A\,P_2$ is negative, therefore the sec $= \frac{r^2}{\cos}$ ought, by the formula, to be negative, which from the figure it appears to be: again, sin $A\,P_3$ is negative, the cos is also negative, hence the tan $= \frac{\sin . r}{\cos}$, ought to be positive, as it is found to be.

We proceed to one of the most important theorems in Trigonometry.

(19.) To find the sine and cosine of the sum and of the difference of two arcs.

Suppose the radius $OA=r$ (fig. 5.), the arc $AB=A$, the arc $BC=B$. Join OB, and draw CI perpendicular to OB, producing it to meet the circumference in G; draw GH, BK, IL, CM, perpendicular, and GP, IQ parallel, to OA, the latter meeting IL, CM in P, Q. By the similar triangles OBK, OIL,

$$IL=\frac{BK\,.\,OI}{OB}=\frac{\sin A\,.\,\cos B}{r},$$

$$OL=\frac{OK\,.\,OI}{OB}=\frac{\cos A\,.\,\cos B}{r}.$$

The triangles CIQ, OBK, having their sides perpendicular each to each, are evidently similar, therefore,

$$CQ=\frac{OK\,.\,CI}{OB}=\frac{\cos A\ \ \sin B}{r},$$

$$IQ=\frac{BK\,.\,CI}{OB}=\frac{\sin A\,.\,\sin B}{r}.$$

Now $IL+CQ=CM=\sin(A+B)$, $OL-IQ=OM=\cos(A+B)$, $IL-IP$ or $CQ=PL=GH=\sin(A-B)$, $OL+PG$ or $IQ=OH=\cos(A-B)$.

Hence,

$$\sin(A+B)=\frac{\sin A\,.\cos B+\cos A\,.\sin B}{r},$$

$$\cos(A+B)=\frac{\cos A\,.\cos B-\sin A\,.\sin B}{r},$$

$$\sin(A-B)=\frac{\sin A\,.\cos B-\cos A\,.\sin B}{r},$$

$$\cos(A-B)=\frac{\cos A\,.\cos B+\sin A\,.\sin B}{r}.$$

In this demonstration we have supposed A and B, and even $A+B$, less than a quadrant; but the same construction and demonstration being applied to arcs of any magnitude, the above formulæ will be found to be general. Thus, when A is supposed greater, and B less, than a quadrant, but such that $A+B$ may be greater than a semicircle, we shall find

$$IL=\frac{BK\,.\,OI}{OB}=\frac{\sin A\,.\cos B}{r},$$

$$CQ=\frac{OK\,.\,CI}{OB}=\frac{-\cos A\,.\sin B}{r}:$$

but, at the same time, we shall have $-CM=IL-CQ$, hence as before,

$$\sin(A+B)=\frac{\sin A\,.\cos B+\cos A\,.\sin B}{r}.$$

We shall leave it to the student to construct the figure for this case, and to show that the other theorems have equal generality.

(20.) It considerably simplifies trigonometrical investigations, and renders formulæ more easy to be remembered, to suppose the radius$=1$; adopting that hypothesis, the preceding formulæ become,

$$\begin{aligned}
\sin(A+B) &= \sin A \, . \cos B + \cos A \, . \sin B,\\
\cos(A+B) &= \cos A \, . \cos B - \sin A \, . \sin B,\\
\sin(A-B) &= \sin A \, . \cos B - \cos A \, . \sin B,\\
\cos(A-B) &= \cos A \, . \cos B + \sin A \, . \sin B.
\end{aligned}$$

(21.) From these, by addition and subtraction we have,

$$\begin{aligned}
\sin(A+B) + \sin(A-B) &= 2 \sin A \, . \cos B,\\
\sin(A+B) - \sin(A-B) &= 2 \cos A \, . \sin B,\\
\cos(A+B) + \cos(A-B) &= 2 \cos A \, . \cos B,\\
\cos(A-B) - \cos(A+B) &= 2 \sin A \, . \sin B.
\end{aligned}$$

(22.) These equations being independent of any particular values of A and B, we may suppose $A+B = A'$, $A-B = B'$, or $A = \frac{A'+B'}{2}$, $B = \frac{A'-B'}{2}$;

then,

$$\sin A' + \sin B' = 2 \sin \frac{A'+B'}{2} \, . \cos \frac{A'-B'}{2},$$

$$\sin A' - \sin B' = 2 \cos \frac{A'+B'}{2} \, . \sin \frac{A'-B'}{2},$$

$$\cos A'+\cos B'=2\cos\frac{A'+B'}{2}\,.\,\cos\frac{A'-B'}{2},$$

$$\cos B'-\cos A'=2\sin\frac{A'+B'}{2}\,.\,\sin\frac{A'-B'}{2}.$$

(23.) In the first two formulæ in (20.), let $B=A$; then,

$$\sin 2A=2\sin A\,.\,\cos A;$$

and,

$$\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A,$$
$$\text{or}=2\cos^2 A-1.$$

Hence,

$$\sin^2 A=\frac{1-\cos 2A}{2};\ \text{and}\ \cos^2 A=\frac{1+\cos 2A}{2}.$$

From these we have $\frac{\sin^2 A}{\cos^2 A}$ or (14.),

$$\tan^2 A=\frac{1-\cos 2A}{1+\cos 2A};\ \text{hence,}\ \cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

(24.) Since $\sin^2 A+\cos^2 A=1$ and $2\sin A\,.\,\cos A=\sin 2A$,

we have,

$$\sin^2 A+\cos^2 A+2\sin A\,.\,\cos A=1+\sin 2A,$$
$$\sin^2 A+\cos^2 A-2\sin A\,.\,\cos A=1-\sin 2A;$$

hence, $\sin A+\cos A=\sqrt{1+\sin 2A},$

$$\sin A-\cos A=\sqrt{1-\sin 2A};$$

therefore,

$$\sin A = \tfrac{1}{2}(\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}),$$

and,

$$\cos A = \tfrac{1}{2}(\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}).$$

(25.) By (21.) we have,

$$\sin A . \sin B = \tfrac{1}{2}\{\cos (A-B) - \cos (A+B)\},$$

let $A+B$ be put for A, and $A-B$ for B, then

$$\sin (A+B) . \sin (A-B) = \tfrac{1}{2}(\cos 2B - \cos 2A)$$
$$= \tfrac{1}{2}(1-2\sin^2 B - 1 + 2\sin^2 A) \text{ (23.)},$$
$$= \sin^2 A - \sin^2 B.$$

Similarly,

$$\cos (A+B) . \cos (A-B) = \tfrac{1}{2}(\cos 2B + \cos 2A)$$
$$= \tfrac{1}{2}(1-2\sin^2 B + 2\cos^2 A - 1) = \cos^2 A - \sin^2 B.$$

(26.) From (21.),

$$\sin (A+B) = 2\sin A . \cos B - \sin (A-B),$$

let $A = nB$, then,

$$\sin (n+1)B = 2\sin nB . \cos B - \sin (n-1)B;$$

making n successively $=0, 1, 2, 3$, &c., we form the following table:

$$\sin B = -\sin - B = \sin B \text{ (9.)};$$
$$\sin 2B = 2\sin B . \cos B;$$
$$\sin 3B = 2\sin 2B . \cos B - \sin B$$
$$= 3\sin B - 4\sin^3 B;$$

$$\begin{aligned}
\sin 4\,B &= 2 \sin 3\,B \,.\, \cos B - \sin 2\,B \\
&= (4 \sin B - 8 \sin{}^3 B) \cos B\,; \\
\sin 5\,B &= 2 \sin 4\,B \,.\, \cos B - \sin 3\,B \\
&= 5 \sin B - 20 \sin{}^3 B + 16 \sin{}^5 B\,; \\
\&\text{c.} &= \&\text{c.}
\end{aligned}$$

(27.) Again,

$$\cos (A+B) = 2 \cos A \,.\, \cos B - \cos (A-B),$$

let $A = n\,B$, then,

$$\cos (n+1)\,B = 2 \cos n\,B \,.\, \cos B - \cos (n-1)\,B\,;$$

making n successively $=0, 1, 2, 3$, &c.,

$$\begin{aligned}
\cos \;\; B &= \cos \quad ; \\
\cos 2\,B &= 2 \cos{}^2 B - 1\,; \\
\cos 3\,B &= 2 \cos 2\,B \,.\, \cos B - \cos B \\
&= 4 \cos{}^3 B - 3 \cos B\,; \\
\cos 4\,B &= 2 \cos 3\,B \,.\, \cos B - \cos 2\,B \\
&= 8 \cos{}^4 B - 8 \cos{}^2 B + 1\,; \\
\cos 5\,B &= 2 \cos 4\,B \,.\, \cos B - \cos 3\,B \\
&= 16 \cos{}^5 B - 20 \cos{}^3 B + 5 \cos B\,; \\
\&\text{c.} &= \&\text{c.}
\end{aligned}$$

(28.) $$\frac{\sin (A+B)}{\sin (A-B)} = \frac{\sin A \,.\, \cos B + \cos A \,.\, \sin B}{\sin A \,.\, \cos B - \cos A \,.\, \sin B}$$

$=\dfrac{\tan A + \tan B}{\tan A - \tan B}$, by dividing the numerator and denominator by cos A . cos B, and observing that $\frac{\sin}{\cos}=\tan$.

Also, similarly dividing by sin A . sin B, and putting cot for $\frac{\cos}{\sin}$, we have

$$\frac{\sin(A+B)}{\sin(A-B)}=\frac{\cot B+\cot A}{\cot B-\cot A}.$$

(29.) $$\frac{\cos(A+B)}{\cos(A-B)}=\frac{\cos A\,.\cos B-\sin A\,.\sin B}{\cos A\,.\cos B+\sin A\,.\sin B}$$

$$=\frac{\cot A-\tan B}{\cot A+\tan B};\text{ or,}=\frac{\cot B-\tan A}{\cot B+\tan A}.$$

(30.) $$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{A+B}{2}\,.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\,.\sin\frac{A-B}{2}}$$

$$=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$$

(31.) $$\frac{\cos B+\cos A}{\cos B-\cos A}=\frac{2\cos\frac{A+B}{2}\,.\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\,.\sin\frac{A-B}{2}}$$

$$=\frac{\cot\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$$

(32.) To find an expression for the tangent of the sum or difference of two arcs.

$$\tan\,(A+B)=\frac{\sin\,(A+B)}{\cos\,(A+B)}$$

$$=\frac{\sin A\,.\cos B+\cos A\,.\sin B}{\cos A\,.\cos B-\sin A\,.\sin B},$$ or (by dividing the numerator and denominator by $\cos A\,.\cos B$),

$$=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}.$$

Similarly,

$$\tan\,(A-B)=\frac{\tan A-\tan B}{1+\tan A\,.\tan B}.$$

(33.) Since $\tan\,(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}$, let $B=A$, then $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$.

(34.) Suppose $A+B+C=\pi$, then $A+B=\pi-C$, and $\tan(A+B)=\tan(\pi-C)$, or

$$\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B}=-\tan C \text{ (13).}$$

hence we have the following remarkable equation,
$\tan A+\tan B+\tan C=\tan A\,.\,\tan B\,.\,\tan C.$

(35.) $\tan(A+B+C)=\dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\,.\,\tan C}$ (32.),

$$=\frac{\tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C}{1-\tan A\,.\,\tan B-\tan A\,.\,\tan C-\tan B\,.\,\tan C}.$$

If $A+B+C=\pi$, $\tan(A+B+C)=0$; therefore $\tan A+\tan B+\tan C-\tan A\,.\,\tan B\,.\,\tan C=0$, which is the equation in (34).

(36.) We have hitherto considered the relations which exist generally between different arcs: as there are some which depend upon the numerical values of the trigonometrical lines belonging to certain arcs, we shall proceed to consider them.

(37.) Since $\sin 45°=\cos 45°$; $\sin^2 45°+\cos^2 45°$, or $2\sin^2 45°=1$, therefore, $\sin 45°=\dfrac{1}{\sqrt{2}}=\cos 45°$.

Also, $\tan 45° = \dfrac{\sin 45°}{\cos 45°} = 1 = \cot 45°$;

$\sec 45° = \dfrac{1}{\cos 45°} = \sqrt{2} = \operatorname{cosec} 45°$.

(38.) Let the angle $A\,O\,P = 60°$ (fig. 6.); then $O\,A\,P + O\,P\,A = \pi - 60° = 120°$, and, being equal to one another, each$=60°$; therefore the triangle $O\,A\,P$ is equilateral, and the perpendicular $P\,M$ bisects $O\,A$ in M: hence

$\cos 60° = \frac{1}{2} = \sin 30°$; also, $\sin 60° = \sqrt{1-\frac{1}{4}} = \dfrac{\sqrt{3}}{2}$ $= \cos 30°$; $\tan 60° = \sqrt{3} = \cot 30°$; $\sec 60° = 2$ $= \operatorname{cosec} 30°$.

(39.) Since 36° or $2 \times 18° =$ complement of 54 or $3 \times 18°$, we have,

$$\sin 2 \times 18° = \cos 3 \times 18°;$$

or, by (26.), (27.),

$$2 \sin 18° \,.\, \cos 18° = 4 \cos^3 18° - 3 \cos 18°;$$

hence, dividing by cos 18°,

$2 \sin 18° = 4 \cos^2 18° - 3 = 1 - 4 \sin^2 18°$.

Let $\sin 18° = x$, then

$$2\,x = 1 - 4\,x^2;$$

the solution of this quadratic gives,

$$x, \text{ or } \sin 18° = \frac{-1+\sqrt{5}}{4} = \cos 72°.$$

Also, $\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ} = \sqrt{\dfrac{10+2\sqrt{5}}{16}}$

$= \frac{1}{2}\sqrt{\dfrac{5+\sqrt{5}}{2}} = \sin 72^\circ$; and, $1 - 2\sin^2 18^\circ$, or, (23.), $\cos 36^\circ = \dfrac{1+\sqrt{5}}{4} = \sin 54^\circ$.

(40.)

$$\sin(45^\circ + A) = \sin 45^\circ \,.\, \cos A + \cos 45^\circ \,.\, \sin A$$

$$= \frac{\cos A + \sin A}{\sqrt{2}} \text{ (37.)};$$

$$\sin(45^\circ - A) = \sin 45^\circ \,.\, \cos A - \cos 45^\circ \,.\, \sin A$$

$$= \frac{\cos A - \sin A}{\sqrt{2}};$$

$$\tan(45^\circ + A) = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \,.\, \tan A} = \frac{1 + \tan A}{1 - \tan A};$$

$$\tan(45^\circ - A) = \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \,.\, \tan A} = \frac{1 - \tan A}{1 + \tan A}.$$

From the latter two equations we have,

$$\tan(45^\circ + A) - \tan(45^\circ - A) = \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A}$$

$= \dfrac{4 \tan A}{1 - \tan^2 A} =$ by (33.), $2 \tan 2A$.

Also, by the second equation in (21.),

$\sin \overline{60^\circ + A} - \sin \overline{60^\circ - A} = 2 \cos 60^\circ \,.\, \sin A = \sin A$.

And,

$$\sin \overline{36^\circ+A}-\sin \overline{36^\circ-A}=2 \cos 36^\circ . \sin A$$
$$=\frac{\sqrt{5}+1}{2} \sin A;$$
$$\sin \overline{72^\circ+A}-\sin \overline{72^\circ-A}=2 \cos 72^\circ . \sin A$$
$$=\frac{\sqrt{5}-1}{2} \sin A.$$

Subtracting the lower from the upper and transposing,

$$\sin \overline{36^\circ+A}+\sin \overline{72^\circ-A}=\sin A+\sin \overline{36^\circ-A}+\sin \overline{72^\circ+A}.$$

Again, from the third equation in (21.),

$$\cos \overline{36^\circ + A} + \cos \overline{36^\circ-A}=2 \cos 36^\circ . \cos A$$
$$=\frac{\sqrt{5}+1}{2} \cos A,$$
$$\cos \overline{72^\circ+A} + \cos \overline{72^\circ-A} = 2 \cos 72^\circ . \cos A$$
$$=\frac{\sqrt{5}-1}{2} \cos A;$$

hence, subtracting and transposing,

$$\cos \overline{36^\circ+A}+\cos \overline{36^\circ-A}=\cos A+\cos \overline{72+A}+\cos \overline{72-A}.$$

(41.) These are the principal formulæ in the arithmetic of sines.

(42.) The trigonometrical lines which have occurred in the preceding theorems are called natural sines, tangents, &c.; and their numerical values (of which we shall treat hereafter) have, with the corresponding arcs, been placed in tables. Owing to the almost universal practice of performing the calculations in which they occur by means of their logarithms, it has been found convenient to have these also arranged in tables. The supposition, however, of radius $=1$, to which the natural sines, &c., are calculated, rendering the sines and cosines of all arcs, and the tangents of arcs less than 45°, fractional, and their logarithms consequently negative, would be extremely inconvenient. It was advisable, therefore, to assume another value for the radius: in the adoption of which the facility of the addition, subtraction, &c., of its logarithm was kept in view; it has hence been agreed to make the radius $=10^{10}$, or ten thousand millions, the logarithm of which is 10, and consequently to increase the logarithms of the natural sines, &c., by 10; since the radius having been augmented ten thousand million times, the sines, tangents, &c., have, in consequence, been proportionally increased. This constitutes the logarithmic table of sines, tangents, &c.

(43.) In employing this table in the computation of the values of trigonometrical formulæ adapted to radius $=1$, the radius (10^{10}) of the table, must generally be introduced; the condition being observed at the time, that the terms of the equation are to be homogeneous, or each of the same dimension. Thus in the equation,

$$\sin A - \sin B \, . \, \sin D - \cos B \, . \, \cos D \, . \, \cos E = 0,$$

in which the terms are respectively, of one, two and three dimensions, the radius disappears, having been assumed equal to unity in the course of the investigation. In order, therefore, to pass from natural sines to the tabular sines, we multiply the first term by r^2 (calling $10^{10} = r$), and the second by r; the equation $r^2 \, . \, \sin A - r \, . \, \sin B \, . \, \sin D - \cos B \, . \, \cos D \, . \, \cos E = 0$, which results, is homogeneous, and adapted to the tabular radius.

SECTION III.

ON SUBSIDIARY ANGLES.

(44.) Def. A *subsidiary angle* is an arbitrary angle employed to facilitate calculation.

(45.) Trigonometrical formulæ are frequently of such a nature, that the employment of logarithms to obtain their numerical values would be more laborious than the simple calculation by natural sines, &c., and even impracticable. It was, in the computation of such formulæ, that subsidiary angles originated. We are enabled, by availing ourselves of the calculations which have been made in forming trigonometrical tables, to transform such formulæ into others adapted for computation by logarithms; and also to facilitate calculations totally unconnected with trigonometry.

Their use will be best shewn by examples.

Example I. Let it be required to calculate x from the equation $x^2 = b^2 + a^2 \sin{}^2\theta$.

The latter part of the equation being put under the form $b^2\ (1 + \frac{a^2}{b^2} \sin{}^2\theta)$, assume $\frac{a}{b} \sin\theta = \tan\phi$, then we shall have,

$$x^2 = b^2\ (1 + \tan{}^2\phi) = b^2\ \sec{}^2\phi = \frac{b^2}{\cos{}^2\phi};$$

hence $x = \dfrac{b}{\cos\phi}$.

Now to find ϕ, we have the equation,

$$\tan\phi = \frac{a}{b} \sin\theta, \text{ or}$$

$\log\tan\phi = \log a + \log\sin\theta - \log b$; and, to determine x, we have,

$$\log x = \log b + 10 - \log\cos\phi.$$

Example II. Let $x = \sqrt{a^2 - b^2}$; a and b being numbers consisting of several figures.

The equation being put under the form

$x = a\sqrt{1-\frac{b^2}{a^2}}$, let $\frac{b}{a} = \cos\theta$; then $x = a\sqrt{1-\cos^2\theta}$ $= a\sin\theta$.

Hence, $\log x = \log a + \log\sin\theta - 10$; the subsidiary angle θ being previously determined from the equation,

$$\cos\theta = \frac{b}{a}, \text{ or}$$

$$\log\cos\theta = \log b + 10 - \log a.$$

Example III. The following expression, $x = a\cos\theta + b\sin\theta$, is not adapted to logarithmic computation.

Let it be transformed into $x = b\,(\frac{a}{b}\cos\theta + \sin\theta)$: suppose $\frac{a}{b} = \tan\phi$; then,

$$x = b\,(\tan\phi\,.\cos\theta + \sin\theta) = b\left(\frac{\sin\phi\,.\cos\theta}{\cos\phi} + \sin\theta\right)$$

$$= \frac{b}{\cos\phi}(\sin\phi\,.\cos\theta + \cos\phi\,.\sin\theta) = \frac{b\sin(\phi+\theta)}{\cos\phi}.$$

Hence, ϕ being found from the equation,

$\log\tan\phi = \log a + 10 - \log b$, we shall have,

$\log x = \log b + \log\sin(\phi+\theta) - \log\cos\phi$.

Other instances of the use of subsidiary angles will occur in the solutions of plane and spherical triangles.

SECTION IV.

PLANE TRIGONOMETRY.

(46.) A plane triangle consists of six parts, viz:—three sides and three angles: if any three of these parts except the three angles be given, the other three become known by the application of principles, which constitute Plane Trigonometry.

(47.) Let $A B C$ (fig. 7.) be a triangle, right angled at C; and let the sides opposite to the angles $A, B, C,$ be denoted by the small letters $a, b, c,$ respectively. From the point A as a centre, with radius$=1$, describe the circle $G H$, and draw $G K$ perpendicular to $A C$.

By the similar triangles $A G K$, $A B C$,

$$A G : G K :: A B : B C,$$

$$\text{or, } 1 : \sin A :: c : a,$$

therefore, $a = c \sin A$.

And $A G : A K :: A B : A C$;

$$\text{or, } 1 : \cos A :: c : b,$$

therefore, $b = c \cos A$.

Dividing the former equation by the latter, we have,

$$\frac{a}{b} = \frac{\sin A}{\cos A} = \tan A.$$

These three equations, with the elementary ones

$A+B=\frac{\pi}{2}$, and $c^2=a^2+b^2$, are sufficient to solve every case of right angled plane triangles. We shall present them in one view, and apply them to a few examples.

$$a=c\sin A\ldots.(1)\qquad \frac{a}{b}=\tan A\ldots..(3)$$

$$b=c\cos A\ldots.(2)\qquad A+B=\frac{\pi}{2}\ldots..(4)$$

$$c^2=a^2+b^2\ldots.(5).$$

Example I. Suppose the base of a right angled triangle to be 540 yards, and the adjacent acute angle 56° 40′; required the hypothenuse, the perpendicular, and the other acute angle.

Here A and b are given, to find a, B, and c.

By equation (2.), $c=\frac{b}{\cos A}$,

$\therefore \log c=\log b+10-\log\cos A$.

$$\begin{array}{lcr}
\log 540\ldots\ldots\ldots & = & 2\cdot7323938 \\
 & & 10 \\
\hline
 & & 12\cdot7323938 \\
\log\cos 56^\circ\,40' & = & 9\cdot7399748 \\
\hline
\log c\ldots\ldots\ldots\ldots & = & 2\cdot9924190
\end{array}$$

Referring to the table of logarithms, $c=982.6955$ yards.

Also from equ. (3.), $a=b\tan A$,

$\therefore \log a = \log . b + \log . \tan A - 10.$

$$\begin{array}{lcr}
\log 540 \ldots\ldots\ldots & = & 2 \cdot 7323938 \\
\log \tan 56^\circ\, 40' & = & 10 \cdot 1819653 \\
\cline{3-3}
 & & 12 \cdot 9143591 \\
 & & 10 \quad\quad\quad\quad \\
\cline{3-3}
\log a \ldots\ldots\ldots\ldots & = & 2 \cdot 9143591
\end{array}$$

By reference to the table, $a = 821 \cdot 03$ yards.

Again, from equ. (4.),

$$B = \frac{\pi}{2} - A = 90^\circ - 56^\circ\, 40' = 33^\circ\, 20'.$$

Example II. Given the hypothenuse $= 58 . 45$, and the base $= 31 . 5$, to find the angles and the perpendicular.

Here $c = 58 . 45$, and $b = 31 . 5$.

By equ. (2.), $\cos A = \frac{b}{c}$,

$\therefore \log \cos A = \log b + 10 - \log c.$

$$\begin{array}{lcr}
\log 31 . 5 \ldots\ldots\ldots & = & 1 \cdot 4983106 \\
 & & 10 \quad\quad\quad\quad \\
\cline{3-3}
 & & 11 \cdot 4983106 \\
\log 58 . 45 \ldots\ldots & = & 1 \cdot 7667845 \\
\cline{3-3}
\log \cos A \ldots\ldots\ldots & = & 9 \cdot 7315261
\end{array}$$

referring to the logarithmic table of sines, we find

$$A = 57^\circ\, 23'\, 23''.$$

Hence, $B=\frac{\pi}{2}-A=32^{\circ}\ 36'\ 37''$.

Also, $a=c\sin A$,

$\therefore \log a=\log c+\log\sin A-10$.

$$\begin{array}{lcr}
\log 58\,.\,45\ldots\ldots\ldots & = & 1\cdot 7667845 \\
\log\sin 57^{\circ}\ 23'\ 23'' & = & 9\cdot 9254955 \\
\hline
 & & 11\cdot 6922800 \\
 & & 10 \\
\hline
\log a\ldots\ldots\ldots\ldots\ldots\ldots & = & 1\cdot 6922800
\end{array}$$

$\therefore a=49\,.\,2364$.

The side a however, may be found independently of the angle A, by equ. (5), $c^2=a^2+b^2$, or, $a^2=c^2-b^2$; and, as the numbers b and c consist of several figures, the calculation may be thus facilitated:

since $c^2-b^2=(c+b)\,.\,(c-b)$, we have

$a=\sqrt{(c+b)\,.\,(c-b)}$; taking the logarithms,

$\log a=\frac{1}{2}\{\log(c+b)+\log(c-b)\}$.

$$\begin{array}{lcr}
\log(c+b)\text{ or }89\,.\,95 & = & 1\cdot 9540012 \\
\log(c-b)\text{ or }26\,.\,95 & = & 1\cdot 4305588 \\
\hline
 & 2) & 3\cdot 3845600 \\
\hline
\log a\text{ or }49\,.\,2364\ldots & = & 1\cdot 6922800\text{, as before.}
\end{array}$$

Example III. Given $a=759\,.\,4$ and $b=33\,.\,29$ to find c.

The equation $c^2=a^2+b^2$ directly determines c, but as the operation is somewhat laborious it may be facilitated in the following manner.

Since $c^2=a^2+b^2=a^2\left(1+\frac{b^2}{a^2}\right)$.

Let $\frac{b}{a}=\tan\theta$, then $c^2=a^2(1+\tan^2\theta)=a^2\sec^2\theta$

$=\frac{a^2}{\cos^2\theta}$;

hence, $c=\frac{a}{\cos\theta}$:

now, $\log\tan\theta=\log b+10-\log a$, from which, by computation, we find $\theta=2^\circ\,30'\,36''$, and thence, $\cos\theta$ $=9.9995833$.

Then $\log c=\log a+10-\cos\theta$: from this $c=760.129$.

(48.) The sides of a plane triangle are proportional to the sines of the opposite angles.

Let $A\,B\,C$ be a plane triangle (fig. 8.), and $B\,D$ the perpendicular from B upon the side $A\,C$; then by (47.),
$B\,D=A\,B\,.\sin A$, also $B\,D=B\,C\,.\sin C$; for if the angle $A\,C\,B$ is obtuse, $\sin B\,C\,D=\sin A\,C\,B$ (13). Hence, generally, in the triangle $A\,B\,C$,

$$A\,B\,.\sin A=B\,C\,.\sin C, \text{ or}$$
$$A\,B : B\,C :: \sin C : \sin A.$$

(49.) To find the cosine of an angle when the three sides are given.

When the triangle ACB is acute angled,

$$AB^2 = AC^2 + BC^2 - 2AC \cdot CD, \text{ (Euc. II. 13)};$$

and, when it is obtuse, C being the obtuse angle,

$$AB^2 = AC^2 + BC^2 + 2AC \cdot CD. \text{ (II. 12.)}$$

In the first case, $CD = BC \cdot \cos C$;

in the second, $CD = BC \cdot \cos BCD$

$$= -BC \cdot \cos ACB \text{ (13.)};$$

hence, generally,

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cdot \cos C.$$

therefore, $\cos C = \dfrac{AC^2 + BC^2 - AB^2}{2AC \cdot BC}$; or, adopting the notation in (47.),

$$\cos C = \frac{b^2 + a^2 - c^2}{2ba}.$$

This form is not adapted to logarithmic computation.

(50.) Resuming the equation, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$,

we have, $1 + \cos C$, or (23.),

$$2\cos^2\frac{C}{2} = 1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a+b)^2 - c^2}{2ab}$$

$$= \frac{(a+b+c)\cdot(a+b-c)}{2ab}.$$

Let $a+b+c=2\,S$, then $a+b-c=2\,(S-c)$,

hence, $2\cos^2\frac{C}{2}=\frac{2\,S\,.\,2\,(S-c)}{2\,a\,b}$, or,

$$\cos^2\frac{C}{2}=\frac{S\,.\,(S-c)}{a\,b}.$$

(51.) Also, $1-\cos C$, or $2\sin^2\frac{C}{2}=1-\frac{a^2+b^2-c^2}{2\,a\,b}$

$$=\frac{c^2-(a-b)^2}{2\,a\,b}=\frac{(c+a-b)\,.\,(c+b-a)}{2\,a\,b}$$

$$=\frac{2\,(S-b)\,.\,2\,(S-a)}{2\,a\,b};$$

hence, $\sin^2\frac{C}{2}=\frac{(S-b)\,.\,(S-a)}{a\,b}$.

(52.) Dividing this equation by the preceding one, we have,

$$\frac{\sin^2\frac{C}{2}}{\cos^2\frac{C}{2}},\text{ or }\tan^2\frac{C}{2}=\frac{(S-a)\,.\,(S-b)}{S\,.\,(S-c)}.$$

(53.) Multiply the product of the same equations by 4, then,

$$4\sin^2\frac{C}{2}\,.\,\cos^2\frac{C}{2},\text{ or (23.),}$$

$$\sin^2 C=\frac{4\,S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)}{a^2\,b^2}.$$

All these forms are convenient for calculation by logarithms.

(54.) Given the two sides (a, b), and the included angle (C), to find the other angles and the third side.

From (47), $a : b :: \sin A : \sin B$;
hence,

$$a+b : a-b :: \sin A+\sin B : \sin A-\sin B,$$

therefore,

$$\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}\ (30).$$

Now, $A+B+C=\pi$, or $\dfrac{A+B}{2}=\dfrac{\pi}{2}-\dfrac{C}{2}$; and

$$\tan\frac{A+B}{2}=\cot\frac{C}{2};$$

$$\text{hence, } \frac{a+b}{a-b}=\frac{\cot\dfrac{C}{2}}{\tan\dfrac{A-B}{2}};$$

therefore,

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

$\dfrac{A+B}{2}$ and $\dfrac{A-B}{2}$ being known, their sum gives the value of A, and their difference that of B.

The third side (c), may be determined by (48).

(55.) It is, however, frequently desirable to obtain the third side without the previous process of finding the angles; and this may be accomplished by the following method.

By (49.),

$$\begin{aligned} c^2 &= a^2+b^2-2\,a\,b\cos C \\ &= a^2+b^2+2\,a\,b-2\,a\,b-2\,a\,b\cos C \\ &= (a+b)^2-2\,a\,b\,(1+\cos C) \\ &= (a+b)^2-4\,a\,b\,\cos^2\frac{C}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} c^2 &= (a-b)^2+2\,a\,b-2\,a\,b\cos C \\ &= (a-b)^2+2\,a\,b\,(1-\cos C) \\ &= (a-b)^2+4\,a\,b\,\sin^2\frac{C}{2}. \end{aligned}$$

Multiply the first equation by $\sin^2\frac{C}{2}$, the second by $\cos^2\frac{C}{2}$, and add the results, we shall then have,

$$c^2\left(\sin^2\frac{C}{2}+\cos^2\frac{C}{2}\right), \text{ or (16.),}$$

$$\begin{aligned} c^2 &= (a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2} \\ &= (a-b)^2\cos^2\frac{C}{2}\left\{1+\left(\frac{a+b}{a-b}\right)^2\tan^2\frac{C}{2}\right\}. \end{aligned}$$

Let θ be a subsidiary angle such that

$$\tan\theta=\frac{a+b}{a-b}\tan\frac{C}{2},$$

then, the expression within the vinculum $=\sec^2\theta=\frac{1}{\cos^2\theta}$; hence by substitution we have,

$$c^2=\frac{(a-b)^2\cos^2\frac{C}{2}}{\cos^2\theta}, \text{ or}$$

$$c=\frac{(a-b)\cos\frac{C}{2}}{\cos\theta}.$$

(56.) Since we have not assumed any particular relation to exist among the sides or angles, the preceding equations, by a slight attention to their form, will easily be adapted to any required part.

Thus in (51.) we have, $\sin^2\frac{C}{2}=\frac{(S-a)\,.\,(S-b)}{a\,b}$;

hence, by analogy, we shall have,

$$\sin^2\frac{A}{2}=\frac{(S-c)\,.\,(S-b)}{c\,b}.$$

(57.) The principles investigated in the foregoing articles, with the elementary equation $A+B+C=\pi$, are sufficient for the solution of all plane triangles, when any three parts are known. The case, however, must be excepted when the three angles only are given; for then the *ratio* of the sides only becomes known (48).

(58.) There are cases to be considered, called the ambiguous cases, in which the results will be doubtful, or apparently doubtful, since the angles A and $\pi - A$ have equal sines. In right angled triangles, the angles must be less than $\frac{\pi}{2}$, and therefore there is no ambiguity. There is likewise no ambiguity in (50.), (51.), (52.), for $\frac{C}{2}$ is less than $\frac{\pi}{2}$. If the angle is found from sin C, in (53.), it will be obtuse or acute, according as c^2 is greater or less than $a^2 + b^2$. In the case of two sides (a, c), and an angle (A) opposite one of the sides being given—if a be greater than c, the angle A is greater than C, therefore C must be less than $\frac{\pi}{2}$, and there is no ambiguity. But if a be less than c, the angle A is less than $\frac{\pi}{2}$, and the angle C may be either greater or less than $\frac{\pi}{2}$; the triangles $A\,C\,B$, $A\,C'\,B$ (fig. 9.) will then equally satisfy the solution ; hence, in this case, and this alone, there is a real ambiguity.

(59.) We shall now proceed to the application of the preceding formulæ.

Example I. The three sides, *a*, *b*, *c*, of a plane triangle are respectively, 735, 942, 1297; required the three angles.

The application of the formulæ in (52.) will, in this case, be the most convenient, as the same numbers are used in the three calculations.

To find *A*.

$$\text{Here, } \tan\frac{A}{2} = \sqrt{\frac{(S-b)\,.\,(S-c)}{S\,.\,(S-a)}},$$

$$\therefore \log\tan\frac{A}{2} = \tfrac{1}{2}\{\log\overline{S-b} + \log\overline{S-c} + 20$$

$$-(\log S + \log\overline{S-a})\}.$$

$$\begin{array}{llll}
735=a & 1487 & 1487 & 1487\\
942=b & 735 & 942 & 1297\\
1297=c & \overline{752=S-a} & \overline{545=S-b} & \overline{190=S-c}\\
\overline{2)2974} & & & \\
\overline{1487=S} & & &
\end{array}$$

log 545...	2 · 7363965	log 1487...	3 · 1723110
log 190...	2 · 2787536	log 752...	2 · 8762178
	20		6 · 0485288
	25 · 0151501		
	6 · 0485288		
	2)18 · 9666213		

log tan $\frac{A}{2}$ ..9 · 4833106,

$$\text{hence } \frac{A}{2} = 16^\circ\ 55'\ 31''$$
$$\underline{\qquad\qquad\quad 2}$$
$$A = 33^\circ\ 51'\ 2''$$

To find B.

$$\log \tan \frac{B}{2} = \tfrac{1}{2}\{\log \overline{S-a} + \log \overline{S-c} + 20 -$$

$$(\log S + \log \overline{S-b})\};$$

45°.

from this we obtain $B=$[illegible]° 33′ 14″.

The angle C may now be found from the equation $C = \pi - (A+B)$; but as no evidence can be drawn from this conclusion that the previous calculation is correct, it will be most satisfactory to deduce the value of C independently of that equation.

To find C.

$$\log\tan\frac{C}{2}=\tfrac{1}{2}\{\log\overline{S-a}+\log\overline{S-b}+20-(\log S+\log\overline{S-c})\}.$$

This gives $C=100^\circ\ 35'\ 44''$.

These values of A, B, C, satisfying the equation $A+B+C=\pi$, may be considered correct.

When the angle A had been determined, B and C might have been found by (48.); but, as this presupposes the correctness of A, and as other logarithms would have to be sought in the table, the preceding method will generally be found to be more expeditious.

Observation:—The equations in (50.), (51.), (52.), (53.), may be rendered more expeditious in calculation by taking the arithmetical complements * of the logarithms to be subtracted, which, by a little

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* Def. The arithmetical complement (co. ar.) of a number is its defect from 10, or from 20, if greater than 10. It is used as an artifice to perform subtraction by addition. Thus to subtract 5·47 from 9·65, we add the co. ar. of 5·47, which is 4·53 to 9·65, the sum 14·18, diminished by 10, which was introduced in taking the co. ar. gives 4·18, the difference required. The co. ar. is generally taken by subtracting the first figure on the right from 10, and the rest from 9.

practice, may be set down with nearly the same facility as the logarithms: we may thus practically dispense with introducing the tabular radius. Applying the formula in (51), which is the one commonly used when only one angle is required, since the numbers composing it generally lie near each other in the table, we have, to find C in the preceding example,

$$\sin\frac{C}{2}=\sqrt{\frac{(S-a)\,.\,(S-b)}{a\,b}}.$$

$$\begin{array}{ll}
\log 735 \text{ co ar}\ldots & 7\cdot 1337127 \\
\log 942 \text{ co ar}\ldots & 7\cdot 0259491 \\
\log 752 \ldots\ldots\ldots & 2\cdot 8762178 \\
\log 545 \ldots\ldots\ldots & 2\cdot 7363965 \\
\hline
 & 2)19\cdot 7722761 \\
\hline
\log\ \sin\dfrac{C}{2}\ \ldots\ldots & 9\cdot 8861380
\end{array}$$

$$\therefore\ \frac{C}{2} = 50^\circ\ 17'\ 52'' \atop \phantom{\frac{C}{2} = 50^\circ\ 17'\ 5}2$$

$$C = \underline{100^\circ\ 35'\ 44''}, \text{ as before.}$$

Example II. Given the two sides of a triangle $=375$ and 814, and the included angle$=74^\circ\ 47'$, to find the other angles and the third side.

Supposing $a=814$, $b=375$, we have $a+b=1189$, $a-b=439$; also $\frac{C}{2}=37^\circ\ 23'\ 30''$.

By (54.), $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

log 439	2·6424645
log cot $37^\circ\ 23'\ 30''$	10·1167206
log 1189 co ar	6·9248181
log tan $\frac{A-B}{2}$	9·6840032

Hence, $\frac{A-B}{2}=25^\circ\ 47'$, also $\frac{A+B}{2}=52^\circ\ 36'\ 30''$, therefore $A=78^\circ\ 23'\ 30''$, and $B=26^\circ\ 49'\ 30''$.

To find the side c we have the proportion

$$\sin B : \sin C :: b : c$$

log sin $26^\circ\ 49'\ 30''$ co ar	0·3455665
log sin $74^\circ\ 47'$	9·9845004
log 375	2·5740313
log c	2·9040982

Hence $c=801·86$.

The side c may be found independently of A or B, by (55.); and, owing to the nature of the quantities concerned, with greater facility: for this purpose we have the equations

$$\tan\theta=\frac{a+b}{a-b}\tan\frac{C}{2},$$

$$c=\frac{(a-b)\cos\frac{C}{2}}{\cos\theta}.$$

log 1189	3·0751819
log tan 37° 23′ 30″ ...	9·8832794
log 439	2·6424645
log tan θ................	10·3159968

$$\therefore\ \theta=64^\circ\,12'\,59''\cdot 6.$$

log cos θ, co ar	0·3615398
log 439	2·6424645
log cos 37° 23′ 30″....	9·9000955
log c....................	2·9040998

$$\therefore\ c=801\cdot 86.$$

The slight difference observed in the values of log c is owing to the imperfection of the tables.

Example III. Suppose the two angles A and B to be 30°, and 42° 14′, and the contained side c $27\frac{1}{2}$; required the other parts.

By the equation $A+B+C=\pi$, we have $C=107^\circ\,46'$, and by (48.) the sides a and b are found to be 14·4386 and 19·4099 respectively.

Example IV. Given the angle $A=23^\circ\,27'$, the side $a=115\cdot08$, and the side $c=75$, to find the rest.

By (48.) $\sin C=\frac{c\sin A}{a}$, the value of C must be taken less than $\frac{\pi}{2}$ (vid. art. 58.) The angle B will now be found from the equation $A+B+C=\pi$, and the side b from (48). Performing the calculations, we find,

$C=15^\circ\,1'\,54''$, $B=141^\circ\,31'\,6''$, $b=179\cdot948$.

If, in this example, c had been greater than a, the angle C might have been taken either greater or less than $\frac{\pi}{2}$ (58). But when the case is applied to the solution of a problem, the nature of the inquiry will remove all ambiguity.

(60.) In the determination of the heights and distances of objects, horizontal angles are observed by the theodolite; and vertical angles either by the theodolite furnished with a vertical arch, or by a quadrant having a plummet suspended from the centre, and furnished with open or telescopic sights.

Lines upon the ground may be measured by a

Gunter's chain, or a common measuring tape: but a better method consists in measuring upon a stretched rope in the direction of the line, by means of a seasoned deal rod twenty feet long. At the end of each rod, a pin may be stuck in the rope during the replacement of the rod. When the rope has to be drawn forward, the position of the end of the last rod must be accurately noted, and the measurement may be carried on as before.

To take the angle of elevation or depression of an object, *B* or *B′* (fig. 10.) by means of a common quadrant, the eye at *E* or *A* respectively, must be directed through the sights, to the object; the plumb-line will then mark along the arch graduated from *D* to *E*, the angle *D A P*; which is evidently the measure of the angle of elevation *B A R*, or of depression *H A B′*.

(61.) When the necessary lines and angles have been measured, others must be calculated from these, so as to lead to the end in view; but as this depends upon the particular problem under consideration, no general rules can be given. The student will therefore have to depend upon his own resources.

Example I. Standing upon a horizontal plain, at the distance of 75 yards from a steeple, I observed the angle of elevation of the top to be 42° 35′; what was the height, the centre of the quadrant being 5 · 75 feet from the ground?

Let A be the centre of the instrument (fig. 11.), and $A\,C$ the horizontal line$=$75 yards; then, by equ. 3, art. 47,

$$B\,C = A\,C \,.\, \tan A.$$

log 75	1 · 8750613
log tan 42° 35′	9 · 9633204
log 68 · 9258	1 · 8383817

$$\begin{aligned} \text{Hence } B\,C &= 68 \cdot 9258 \\ A\,D &= \underline{\ 1 . 9166\ } \\ \text{Height } &\ \ 70 \cdot 8424 \text{ yards.} \end{aligned}$$

Example II. Being desirous to know the distance of a remarkable object, I measured a line of 300 yards, and observed at each extremity the angles which the object made with this line to be 74° 48′ and 65° 19′; what was my distance from it?

Suppose $A\,C$ (fig. 12.) to be the measured line $=$300 yards, B the distant object, also A and C the observed angles$=$74° 48′ and 65° 19′ respectively;

No 1

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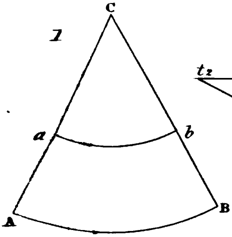

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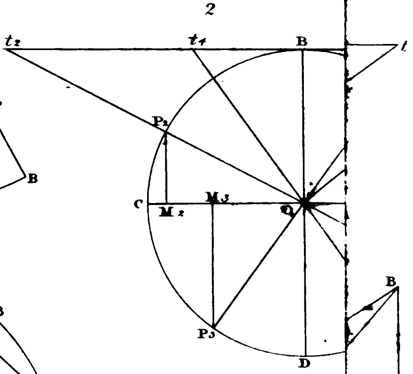

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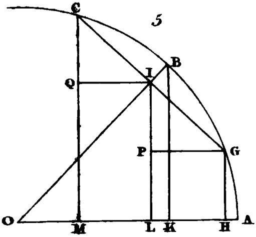

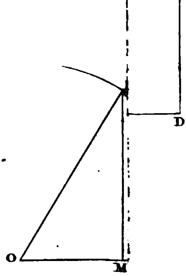

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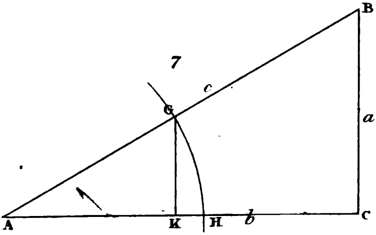

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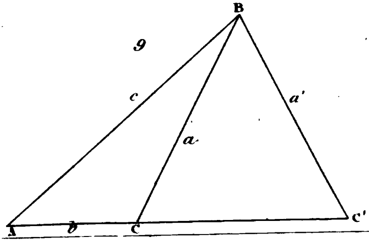

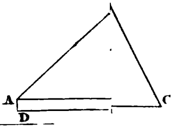

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then if AB be the required distance, we shall obtain it by the following proportion.

$$\sin B : \sin C :: AC : AB.$$

log 300	2 · 4771213
log sin 65° 19′	9 · 9583869
log sin 39° 53′	9 · 8070114
log AB	2 · 6284968

Hence $AB = 425 \cdot 1065$ yards

Example III. To find the distance between two inaccessible objects, B and C (fig. 13.), we must measure a base as AD, and observe the angles at each extremity which both objects make with this line. Suppose we have found $AD = 743 \cdot 55$ yards, $BAD = 113^\circ\ 44'$, $CAD = 48^\circ$, $ADC = 104^\circ\ 13'$, and $BDA = 37^\circ\ 8'$; we shall then get the angles $ABD = 29^\circ\ 8'$ and $ACD = 27^\circ\ 47'$. Hence we shall have AB and AC by these proportions,

$$\sin ABD : \sin ADB :: AD : AB$$
$$\sin ACD : \sin ADC :: AD : AC$$

performing the computations, $AB = 921 \cdot 980$ and $AC = 1546 \cdot 30$. Now, in the triangle ABC, the two sides AB, AC are known, with the included angle $BAC = DAB - DAC$* $= 65^\circ\ 44'$, to find BC.

By (54.), $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

$$= \frac{624\cdot 32}{2468\cdot 28}\cot 32^\circ\ 52'.$$

log 624 · 32	2 · 7954072
log cot 32° 52, ...	10 · 1896975
	12 · 9851047
log 2468 · 28............	3 · 3923944
log tan $\frac{B-C}{2}$	9 · 5927103

$$\text{Hence, } \frac{B-C}{2} = 21^\circ\ 22'\ 46''$$

$$\text{But ... } \frac{B+C}{2} = 57^\circ\ \ 8'$$

$$\text{Therefore } B = 78^\circ\ 30'\ 46''$$

$$\text{and } C = 35^\circ\ 45'\ 14''$$

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* We have supposed the four points A, B, C, D, to lie in the same plane; when this condition does not hold, the angle BAC will no longer be the difference between DAB and DAC: in this case it will be necessary to have the value of that angle by direct measurement; in other respects the process will be the same.

Now to find $B\,C$ we have,

$$\sin A\,C\,B : \sin B\,A\,C :: A\,B : B\,C.$$

log 921·98	2·9647217
log 65° 44′..............	9·9598246
	12·9245463
log sin 35° 45′ 14″	9·7666394
log $B\,C$..................	3·1579069

Hence the required distance $B\,C = 1438\cdot 5$ yards.

Example IV. Three points A, B, C (fig. 14.) in the map of a country being given, it is required to determine the position of a fourth point S, and the three distances $A\,S$, $B\,S$, $C\,S$; the angle $A\,S\,B$, $B\,S\,C$ being known, and the four points lying in the same plane.

Describe a circle about the triangle $A\,C\,S$, which will intersect $B\,S$ in D; join $A\,D$, $D\,C$. Since angles in the same segment of a circle are equal, $D\,A\,C = D\,S\,C$ and $A\,C\,D = A\,S\,D$. Hence in the triangle $A\,D\,C$, the angles A and C are known and the side $A\,C$, therefore $A\,D$ may be found. In the triangle $A\,B\,C$ the three sides are known, hence the angle $B\,A\,C$ may be determined, and thence $B\,A\,D$. Now, in the triangle $A\,B\,D$, the two sides $A\,B$, $A\,D$

are given with the included angle, to find the angle ABD: then the angles ASB, ABS, and the side AB being known, the distances AS, BS may be found. Lastly, in the triangle ACS all the angles and the side AC are given, to find CS.

The problem may be solved geometrically by describing upon AB a segment of a circle containing an angle equal to ASB, and upon BC a segment containing one equal to BSC. The intersection of the arcs will determine the position of the point S.

(62.) To find the area of a triangle which has the three sides given.

Of the triangle ABC (fig. 8.),

$$\text{area} = \frac{AC \cdot BD}{2}$$

$$= \frac{AC \cdot AB \cdot \sin A}{2}, \text{ or by (53.)}$$

$$= \sqrt{S \,.\, (S-a) \,.\, (S-b) \,.\, (S-c)}.$$

Hence,

$$\log \text{area} = \log \tfrac{1}{2} AC + \log AB + \log \sin A - 10.$$

$$\text{or} = \tfrac{1}{2} (\log S + \log \overline{S-a} + \log \overline{S-b} + \log \overline{S-c}).$$

(63.) To find the radius of the circle inscribed in a given triangle.

Let $A\,B\,C$ be the triangle (fig. 15.); bisect the angles $B\,A\,C$, $A\,C\,B$ by the straight lines $A\,O$, $C\,O$; the point of intersection O, will be the centre of the inscribed circle (Euc. IV. 4). Draw $O\,R$, $O\,P$, $O\,Q$, perpendicular to the sides of the triangle; join $O\,B$, and suppose r to denote the required radius: then,

$$\text{Area of } \triangle\, B\,O\,C = \frac{B\,C}{2} \cdot O\,P = \frac{a}{2} \cdot r,$$

$$A\,O\,C = \frac{A\,C}{2} \cdot O\,Q = \frac{b}{2} \cdot r,$$

$$A\,O\,B = \frac{A\,B}{2} \cdot O\,R = \frac{c}{2} \cdot r;$$

By addition, we have,

$$\text{area of } \triangle\, A\,B\,C = \frac{a+b+c}{2} r = S\,r,$$

$$\text{or, } \sqrt{S \cdot (S-a) \cdot (S-b) \cdot (S-c)} = S\,r,$$

$$\therefore r = \sqrt{\frac{(S-a) \cdot (S-b) \cdot (S-c)}{S}}.$$

(64.) To find the radius of the circumscribed circle.

Suppose $A\,B\,C$ (fig. 16.) to be the triangle; bisect the sides $A\,C$, $B\,C$ in L, M; and draw the lines $L\,O$, $M\,O$, perpendicular to them: O will be the centre of the circumscribed circle. (Euc. III. 25.)

Join $A\,O$, $B\,O$, $C\,O$, and let each be denoted by R; then,

$$\text{area of } \triangle A\,B\,C = \frac{A\,B\,.\,A\,C}{2}\,.\,\frac{\sin B\,A\,C}{R} \quad (62.)$$

$$= \frac{b\,c}{2}\,.\,\frac{\sin \dfrac{B\,O\,C}{2}}{R};$$

$$\text{but, } \sin \frac{B\,O\,C}{2} = \frac{\text{chd } B\,O\,C}{2} = \frac{a}{2}; \text{ hence,}$$

$$\text{area of } \triangle\ A\,B\,C = \frac{a\,b\,c}{4\,R};$$

$$\text{therefore } R = \frac{a\,b\,c}{4 \text{ area } \triangle\ A\,B\,C}$$

$$= \frac{a\,b\,c}{4\sqrt{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)}}.$$

$$\text{Hence, also } R\,.\,r = \frac{a\,b\,c}{4\,S} = \frac{a\,b\,c}{2\,(a+b+c)}.$$

(65.) To find the area of the quadrilateral figure that can be inscribed in a circle.

Let the sides $A\,B$, $B\,C$, $C\,D$, $A\,D$ (fig. 17.) be denoted by a, b, c, d: also let the angle $A B C = \theta$, $\therefore A D C = \pi - \theta$ (Euc. III. 22.) Join $A C$.

The area of the quadrilateral $A B C D$

$$= \frac{a b}{2} \sin \theta + \frac{c d}{2} \sin (\pi - \theta) \text{ (62.)}$$

$$= \frac{a b + c d}{2} \sin \theta.$$

Now, by (49.), $a^2 + b^2 - 2\, a b \cos \theta = A C^2$ $= c^2 + d^2 - 2\, c d \cos (\pi - \theta) = c^2 + d^2 + 2\, c d \cos \theta$;

hence we have, $\cos \theta = \dfrac{a^2 + b^2 - c^2 - d^2}{2 (a b + c d)}$;

but, $\sin^2 \theta = 1 - \cos^2 \theta = (1 + \cos \theta) \,.\, (1 - \cos \theta)$

$$= \left(\frac{2\, a b + 2\, c d - (a^2 + b^2) + c^2 + d^2}{2 (a b + c d)}\right)$$

$$. \left(\frac{2\, a b + 2\, c d + a^2 + b^2 - (c^2 + d^2)}{2 (a b + c d)}\right)$$

$$= \frac{(c + d)^2 - (a - b)^2}{2 (a b + c d)} \,.\, \frac{(a + b)^2 - (c - d)^2}{2 (a b + c d)}$$

$$= \frac{(b + c + d - a) \,.\, (a + c + d - b) \,.\, (a + b + d - c) \,.\, (a + b + c - d)}{4 (a b + c d)^2};$$

suppose $a + b + c + d = 2\, S'$; and we then have,

$$\sin^2 \theta = \frac{4\, (S' - a) \,.\, (S' - b) \,.\, (S' - c) \,.\, (S' - d)}{(a b + c d)^2}$$

hence, $\sin\theta$

$$=\frac{2}{a\,b+c\,d}\sqrt{(S'-a)\,.\,(S'-b)\,.\,(S'-c)\,.\,(S'-d)};$$

therefore, by substitution in the foregoing expression for the area we obtain,

area $A\,B\,C\,D$

$$=\sqrt{(S'-a)\,.\,(S'-b)\,.\,(S'-c)\,.\,(S'-d)}.$$

The value of the diagonal $A\,C$ expressed in terms of the four sides may readily be obtained.

We have, $A\,C^2=a^2+b^2-2\,a\,b\cos\theta$

$$=a^2+b^2-2\,a\,b\left(\frac{a^2+b^2-c^2-d^2}{2\,(a\,b+c\,d)}\right)$$

$$=\frac{c\,d\,(a^2+b^2)+a\,b\,(c^2+d^2)}{a\,b+c\,d}$$

$$=\frac{(a\,d+b\,c)\,.\,(a\,c+b\,d)}{a\,b+c\,d}$$

$$\therefore\ AC=\sqrt{\frac{(a\,d+b\,c)\,.\,(a\,c+b\,d)}{a\,b+c\,d}}.$$

(66.) To find the area of a regular polygon.

Let A and B (fig. 18.) be two contiguous angles of the polygon, and let them be bisected by the straight lines $A\,O$, $B\,O$; then the polygon will evidently be composed of n triangles, each equal to

AOB; also let $AB=a$, and draw OP perpendicular to AB.

The area of the triangle $AOB=AP\,.\,OP$

$$=\frac{a}{2}\cdot\frac{a}{2}\tan OAP=\frac{a^2}{4}\cot AOP=\frac{a^2}{4}\cot\frac{\pi}{n}:$$

hence,

$$\text{area of the polygon}=\frac{na^2}{4}\cot\frac{\pi}{n}.$$

or,

$$\log \text{area}=\log n-\log 4+2\log a+\log\cot\frac{\pi}{n}-10.$$

Example. Let $a=1$, $n=7$, then $\frac{\pi}{n}=25^\circ\,42'\,51''\frac{3}{7}$,

hence, log area of heptagon

$$=\log 1\cdot 75+\log\cot 25^\circ\,42'\,51''\tfrac{3}{7}-10.$$

log 1·75	0·2430380
log cot 25° 42′ 51″$\frac{3}{7}$.....	10·3173364
log area.....................	0·5603744

$\therefore$ Area $=3\cdot 633912$.

(67.) We shall conclude this section with the following examples for exercise in plane trigonometry.

1. At the distance of 170 feet from the bottom of a tower, the angle of elevation of the top was observed with a quadrant to be 52° 30′. What was the height of the tower?

Answer, 221·55 feet.

2. Required the perpendicular height of a hill, the angle of elevation taken at the bottom of it being 46°, and 200 yards further off, on a level with the bottom, the angle was 31°.

Answer, 286·28 yards.

3. In any right angled triangle *A B C*, *C* being the right angle,

$$\frac{c}{a} = \sec B, \qquad \frac{c}{b} = \sec A.$$

4. What was the perpendicular height of a balloon, when the angles of elevation, taken by two observers at the same time on opposite sides of it, and in the same vertical plane, were 65° 55′ and 71° 11′; the observers being separated 470 yards?

Answer, 596·655 yards.

5. The two sides of a plane triangle are 718·48

and 519·77 feet, and the included angle is 24° 37′; required the area.

Area$=$77778·2 square feet.

6. Prove that $\tan^2 A - \sin^2 A = \tan^2 A \cdot \sin^2 A$.

7. Given $\tan^2 x = 1\frac{2}{5}$, to determine x.

$x = 49° \, 47' \, 50''$.

8. The three sides of a plane triangle are 386, 518 and 179 yards, required the area, and the radius of the circumscribing circle.

Area$=$26782½ square yards; radius$=$334·085 yards.

9. From a ship at sea I observed a headland to bear E. by S., and after sailing N.E. 12 miles, it bore S.E. by E. Required the distance of the last place of observation from the headland.

Answer, 26 miles.

10. From a window near the bottom of a house which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple and found it to be 40°; then, from another window 18 feet directly above the former,

the angle was 37° 30′; what was the height and distance of the steeple?

Answer, Height = 210·44 feet. Distance = 250·79 feet.

11. Wanting to know the breadth of a river, I measured a base of 500 yards, along the edge of the water, and observed the angles at each end of this line which a tree upon the opposite bank subtended with it; these were 53° and 79° 12′: what was the breadth of the river opposite to the tree?

Answer, 529·48 yards.

12. From two remarkable headlands, I measured to a point inland, 735 yards and 840 yards; the angle at this point subtended by them was observed with a theodolite to be 55° 40′; what was the distance between the headlands?

Answer, 741·2 yards.

13. Being desirous to know the distance between two objects, A and B, I measured a line, CD, upon level ground = 300 yards, and observed the angle $BCD = 58° 20'$, $ACB = 37°$, $ADC = 53° 30'$, and $ADB = 45° 15'$. Required the distance AB.

Answer, $AB = 479 \cdot 79$ yards.

14. The height of the Peak of Teneriffe is $2\frac{1}{3}$ miles; suppose the angle taken at the top of it, as formed by a plumb-line and a line conceived to touch the earth in the horizon, or furthest visible point to be 88° 2′. It is required from these measures to determine the diameter of the earth.

Answer, 7916 miles.

15. From the top of Flamborough-head lighthouse, the angle of depression of a ship at anchor was 3° 38′, and at the bottom of the lighthouse, the angle was 2° 43′; required the horizontal distance of the vessel, and the height of the promontory above the level of the sea; the lighthouse being 85 feet high.

Answer, distance = 5296·4 feet. Height = 251·31 feet.

16. Required the arc, the tangent of which added to the cotangent may be equal to four times the radius.

Answer, 75°.

17. Wanting to know the angle *A C B* formed by the two walls *A C* and *B C*, I measured *A C* = 73·5 yards, *B C* = 84 yards, and *A B* = 74·1 yards.

Answer, *A C B* = 55° 40′.

18. Three places, A, B, C, situated on a level, are distant from each other as follows: $AB = 2\cdot 6625$ miles, $AC = 5\cdot 3$ miles, and $BC = 3\cdot 275$ miles. From a signal S the horizontal angles ASB and BSC are observed to be 13° 30′ and 29° 50′; required the distances of the three places from the signal,—B being nearest to it and appearing between the others A and C.

Answer, $AS = 7\cdot 5714$ miles, $BS = 5\cdot 37103$, and $CS = 6\cdot 55294$ miles.

SECTION V.

SPHERICAL GEOMETRY.

(68.) Def. A *sphere* is a solid bounded on all sides by a curve surface, every point of which is equally distant from a point within, called the *centre*.

A sphere may be conceived to be generated by a semicircle's revolving about its diameter, which remains fixed.

(69.) A straight line from the centre to any point of the surface is called a *radius;* if extended both ways from the centre and terminated by the surface, a *diameter.*

(70.) The section formed by a plane cutting the sphere, is a circle.

This is evident when the plane passes through the centre; since every line drawn from the centre in this plane to the boundary of the section is equal to the radius of the sphere. The section thus made is called a *great circle.*

When the section does not pass through the centre, as AB (fig. 19.); from the centre O, draw OC perpendicular to the plane AB, meeting it in C; take any two points D, E in the boundary of the section, and join OD, OE, CD, CE. Since OC is perpendicular to the plane AB, it is also perpendicular to the straight lines CD, CE drawn in this plane; therefore OCD, OCE are right angles; hence $CD = \sqrt{OD^2 - OC^2}$, and $CE = \sqrt{OE^2 - OC^2}$; but $OD = OE$, being radii of the sphere, therefore $CD = CE$, and hence the section is a circle. This is called a *small circle.*

(71.) Two great circles bisect one another, for their common intersection is the diameter of the sphere bisecting both circles.

(72.) The *pole* of a great or small circle is a point on the surface of the sphere equally distant from the circumference.

Let *G H* be a great circle, *K, L,* any two points in the circumference; draw *P O S* through the centre perpendicular to the plane of the circle, and intersecting the sphere in *P, S*: also join *O K, O L,* and imagine *P K, P L,* to be joined by straight lines. The triangles *O P K, O P L,* are manifestly equal in all respects; hence $PK = PL$, therefore *P* is the pole of the circle *G H*.

In the same way it may be shewn that the point *P* is the pole of the small circle *A B,* which is parallel to *G H*.

Every circle upon the sphere has two poles, for the same demonstration will evidently apply to the point *S*: the word pole, however, in the case of the small circle, is usually restricted to the nearer point *P*.

(73.) A great circle may be drawn through any two points on the surface of the sphere, but not generally through more than two: for the plane of the circle passes likewise through the centre of the sphere, and a plane may pass through any three points, but not generally through more than three.

(74.) Suppose a great circle *P K S,* to pass through the poles, P, S, of the great circle KL; then, since KOP is a right angle, PK is a quadrant; hence the pole of a great circle is situated upon the surface of the sphere at the distance of a quadrant from every point of the circumference.

(75.) A *spherical angle* is formed on the surface of the sphere by the intersection of two great circles; it is measured by the inclination of the planes of those circles: when the planes are perpendicular to each other, the angle is a *spherical right angle.*

Hence the arc KL, which measures the angle KOL, the inclination of the planes of the circles PKS, PLS, will be the measure of the spherical angle P.

(76.) A *spherical triangle* is a portion of the surface of the sphere formed by the three arcs of great circles.

When one of the angles of the triangle is a right angle, it is called *a right angled spherical triangle;* when one of the sides is a quadrant, it is called *quadrantal;* otherwise, it is denominated an *oblique spherical triangle.*

(77.) Two sides of a spherical triangle are greater than the third.

Let *A C B* (fig. 20.) be a spherical triangle, *O* the centre of the sphere; join *A O*, *B O*, *C O*. The solid angle at *O* is formed by the three plane angles *A O B*, *A O C*, *B O C*, any two of which are greater than the third (Euc. XI. 20.): hence any two of the sides *A B*, *A C*, *B C*, which are the measures of these angles, are greater than the third.

(78.) The three sides of a spherical triangle are together less than the circumference of a great circle. For the three plane angles *A O B*, *A O C*, *B O C*, are less than 2π (Euc. XI. 21.); hence the three arcs *A B*, *A C*, *B C*, are less than the circumference of a great circle.

(79.) The surface, *P K S L P* (fig. 19.), included between the semi-circumferences *P K S*, *P L S*, is proportional to the angle *K P L*. For conceive the semicircle *P L S* to remain fixed, whilst *P K S* revolves about the diameter *P S*; it will appear evident that the generated surface, *P K S L P*, will be to the surface of the sphere as the arc *K L* is to the circumference *G H*, or as *K P L* is to 2π. Let r=radius of the sphere, then the surface of the sphere being $=4\pi r^2$,* we have,

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* Differential Calculus.

$$PKSLP : 4\pi r^2 :: P : 2\pi;$$

therefore,

$$\text{surface } PKSLP = 2\,r^2 \times P.$$

(P being taken to represent the arc corresponding to the angle at P, to radius 1).

(80.) Let ABC (fig. 21.) be a spherical triangle, produce the sides to form complete circles, and let the great circle, $DEFGHK$, intersect these circles without the triangle ABC. Since DEG = semicircumference $= EGH$, therefore the arc $DE = GH$; also $ECH = CHL$, therefore $CE = HL$; and $DCG = CGL$, hence $DC = GL$. The triangles DCE, GLH have, therefore, their sides equal, each to each; their corresponding angles are likewise respectively equal, being measured by the inclinations of the planes of the circles. Hence they are equivalent in surface.

Therefore, $CGH + DCE$ or $GLH = 2\,r^2 \times C$,
similarly, $BDK + BFG$ .... $= 2\,r^2 \times B$,
and $AEF + AKH$ .... $= 2\,r^2 \times A$;

by addition, we have,

$$\text{surface of hemisphere } + 2\,ABC = 2\,r^2 (A + B + C),$$

$$\text{or, } 2\,\pi\, r^2 + 2\,ABC = 2\,r^2\,(A + B + C);$$

$\therefore$ surface of triangle $A B C = r^2 (A+B+C-\pi)$: $A+B+C-\pi$, being the excess of the sum of the angles above two right angles, is generally denominated *the spherical excess.*

(81.) Suppose $A B C$ (fig. 24.) to be a spherical triangle, and with the points A, B, C, as poles, let the great circles $E F$, $D F$, $D E$, be described; a triangle, $D E F$, will be formed, which is called the *polar or supplemental triangle.*

Because A is the pole of $E F$, the arc joining A, F is a quadrant; and since B is the pole $D F$, the arc joining B, F is a quadrant: hence the point F is the pole of $A B$. Similarly, D is the pole $B C$, and E that of $A C$. Therefore $A B C$ is the polar triangle to $D E F$.

Call $A B C$ the primitive triangle, and produce the sides $C A$, $C B$, to meet $D E$ in H, K. Now, D and E being the poles of $C B K$ and $C A H$, the arcs $D K$, $E H$, are each = quadrant, hence $D E = D K + E H - K H = \pi - C$. Similarly, $D F = \pi - B$, and $E F = \pi - A$. Therefore the sides of the polar triangle = supplements of the angles of the primitive triangle.

Produce $A C$ to meet $E F$ in L; then $A C = A L + H C - H L = \pi - E$. Similarly, $A B = \pi - F$ and $B C = \pi - D$. Therefore, the sides of the primitive triangle = supplements of the angles of the polar triangle.

(82.) Let a represent the sum of the sides of the polar triangle, and A the sum of the angles of the primitive triangle; then by (81.), $a+A=3\pi$; now, a must be of some magnitude, but less than 2π, by (78.): hence A must be less than 3π, and greater than π.

(83.) Imagine $A B C$ (fig. 20.) to be a spherical triangle, right angled at C. Produce $A B$, $C B$, to D, E, making $A D$ and $C E$ each $=\frac{\pi}{2}$; through D and E describe the great circle $E D F$, meeting $A C$ produced in F. The triangle $E B D$ is called the *complemental triangle.* Since $E C=\frac{\pi}{2}$, and perpendicular to $A C$, the point E is the pole of $A C$ (74.); therefore the angle F is a right angle, and the arc joining A and E a quadrant: hence, A is the pole of $E F$, the angle $A D E$ is a right angle, and $A F$ is a quadrant. Also, since $E F=\frac{\pi}{2}$, $E D$ is the complement of $F D$, or angle A; the angle E or

C F, is the complement of *A C*; *B E* is the complement of *C B*, and *B D* the complement of *A B*.

(84.) The foregoing principles being understood, we shall proceed to the solution of spherical triangles, the general problem being as in plane triangles, from any three known parts to determine the rest.

SECTION VI.

SPHERICAL TRIGONOMETRY.

(85.) The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.

Let *A B C* (fig. 22.) be a spherical triangle, *O* the centre of the sphere; join *A O*, *B O*, *C O*; also draw *C D* perpendicular to the plane *A O B*, *C E* perpendicular to *A O*, and *C F* to *B O*; and join *D E*, *D F*. Since *C D* is perpendicular to the plane *A O B*, the angle *C D E* is a right angle, therefore,

$$CE^2+OE^2=OC^2=OD^2+CD^2$$
$$=OD^2+CE^2-ED^2 \therefore OE^2=OD^2-ED^2;$$

hence the triangle $O\,D\,E$ is right angled, or $D\,E$ is perpendicular to $A\,O$; but $C\,E$ is likewise perpendicular to $A\,O$, therefore the angle $C\,E\,D$ is the measure of the spherical angle A; also, $C\,E$ is the sine of $A\,C$.

Hence,

$$C\,D = C\,E \,.\, \sin C\,E\,D = \sin A\,C \,.\, \sin A:$$

similarly,

$$C\,D = \sin B\,C \,.\, \sin B;$$

therefore,

$$\sin A\,C \,.\, \sin A = \sin B\,C \,.\, \sin B,$$

or,

$$\sin A\,C : \sin B\,C :: \sin B : \sin A.$$

(86.) Let $A\,B\,C$ (fig. 20.) be a spherical triangle right angled at C, and $E\,B\,D$ the complemental triangle (83.); also let the sides $B\,C$, $A\,C$, $A\,B$, be denoted by a, b, c, respectively: then by (85.),

$$\sin D : \sin E :: \sin B\,E : \sin B\,D, \text{ or,}$$

$$1 : \cos b :: \cos a : \cos c.$$

$$\therefore \cos c = \cos a \,.\, \cos b \ldots\ldots. (1)$$

Also,

$$1 : \sin A :: \sin c : \sin a.$$

$$\therefore \sin a = \sin c \,.\, \sin A \ldots\ldots (2)$$

Similarly,

$$\sin b = \sin c \,.\, \sin B \ldots\ldots (3)$$

In the triangle *E B D*,

$$1 : \sin B :: \sin BE : \sin DE, \text{ or}$$

$$1 : \sin B :: \cos a : \cos A.$$

$$\therefore \cos a = \frac{\cos A}{\sin B} \ldots\ldots (4)$$

Similarly, $\cos b = \dfrac{\cos B}{\sin A} \ldots\ldots (5)$

Multiplying these two equations together,

$\cos a \,.\, \cos b$ or $\qquad \cos c = \cot A \,.\, \cot B \ldots\ldots (6)$

From this principle,

$\cos BE = \cot B \,.\, \cot E$ or $\sin a = \tan b \,.\, \cot B \ldots (7)$

Similarly, $\sin b = \tan a \,.\, \cot A \ldots\ldots (8)$

By this, $\sin DE = \tan BD \,.\, \cot E$ or

$$\cos A = \cot c \,.\, \tan b \ldots\ldots (9)$$

Similarly, $\cos B = \cot c \,.\, \tan a \ldots\ldots (10)$

The ten preceding equations comprehend all the cases of right angled spherical triangles: we shall present them in a tabular form.

$\cos c = \cos a \,.\, \cos b$ …… 1.
$\sin a = \sin c \,.\, \sin A$ …… 2.
$\sin b = \sin c \,.\, \sin B$ …… 3.
$\cos A = \cos a \,.\, \sin B$ …… 4.
$\cos B = \cos b \,.\, \sin A$ …… 5.
$\cos c = \cot A \,.\, \cot B$ …… 6.
$\sin a = \tan b \,.\, \cot B$ …… 7.
$\sin b = \tan a \,.\, \cot A$ …… 8.
$\cos A = \tan b \,.\, \cot c$ …… 9.
$\cos B = \tan a \,.\, \cot c$ …… 10.

(87.) In viewing the equations 1, 2, 3, 4, 5, we perceive, on the one hand, the two sides, a, b, affected with sine, and the two angles A, B, and the hypothenuse c, with cosine; on the other hand, a, b, are affected with cosine, and A, B, and c, with sine. We see further, that in each equation, the three parts concerned have the same relative situation that they have in the triangle. For example, a and b join together in the figure[*], and c is separated from them: this relative situation takes place in the first equation.

Of the equations 6, 7, 8, 9, 10, the left side of each is the same as the corresponding one of the first five; on the right hand, *a*, *b*, are affected with tangent, and *A*, *B*, *c*, with cotangent. We also observe, that in each equation, the three parts concerned join together in the triangle, the middle one being that which is separated from the other two in the equation.

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[*] The right angle C being known, is not considered to disjoin the sides a and b.

From these remarkable local properties, Napier invented a kind of technical memory to embrace all the equations, called the *Five Circular Parts.*

The five circular parts are the two sides, the complements of the two angles, and the complement of the hypothenuse. If the three parts concerned in any solution, viz., the two given parts and the required part, join together, that which is in the middle is called the *middle part,* and the other two the *conjoined parts.* But if the three parts do not all join together, the part which is separated, is called the middle part, and the others, the *disjoined parts.* This being settled, the two following rules will manifestly comprise all the cases of right angled spherical triangles.

"*Product of radius and sine of the middle part =product of tangents of the conjoined parts,*" and "*product of radius and sine of the middle part=product of cosines of the disjoined parts.*"

These rules were considered by Professor Playfair to be the happiest example of artificial memory that is known (*b*).

Example I. Suppose the two angles A and B of a right angled spherical triangle to be 72° 18′ and 63° 28′; required the two sides and the hypothenuse.

To find the side a, we take the complement of A, or $\left(\frac{\pi}{2} - A\right)$, for the middle part, and a and the complement of B, or $\left(\frac{\pi}{2} - B\right)$, for the disjoined parts: hence, by the second rule,

$$r \,.\, \cos A = \cos a \,.\, \sin B.$$

$$\therefore \log \cos a = \log \cos A + 10 - \log \sin B.$$

$10 + \log \cos 72° \, 18'$ ..... $19 \cdot 4829208$

$\log \sin 63° \, 28'$ .... $9 \cdot 9516651$

$\log \cos a$ .... $9 \cdot 5312557$

Hence the side $a = 70° \, 8' \, 2''$.

To find the side b, we shall similarly have,

$$\log \cos b = \log \cos B + 10 - \log \sin A.$$

10 + log cos 63° 28′ ..... 19·6500338
log sin 72° 18′ .... 9·9789386

log cos b ... . 9·6710952

Hence $b = 62°\ 2'\ 10''$.

To find the hypothenuse c, we have $\left(\frac{\pi}{2} - c\right)$ for the middle part, and $\left(\frac{\pi}{2} - A\right)$ and $\left(\frac{\pi}{2} - B\right)$ for the conjoined parts; hence, by the first rule,

$$r \,.\, \cos c = \cot A \,.\, \cot B, \text{ or}$$

$$\log \cos c = \log \cot A + \log \cot B - 10.$$

log cot 72° 18′ .... 9·5039822
log cot 63° 28′ .... 9·6983687

19·2023509
10

log cos c .... 9·2023509

$\therefore c = 80°\ 49'\ 51''$.

Example II. The base of a right angled spherical triangle = 34° 10′, and the opposite angle = 71° 15′; required the other parts.

Answer, $A = 22° 51' 38''$, $a = 13° 19' 15''$ and $c = 36° 22' 32''$.

Example III. The two sides are 15° 31′, and 45° 2′, what are the values of the two angles and hypothenuse?

The two angles are 21° 25′ 32″ and 75° 2′ 23″, and the hypothenuse 47° 4′ 59″.

(88.) To find the cosine of an angle when the three sides of the triangle are given.

Let $A B C$ (fig. 23.) be any spherical triangle, O the centre of the sphere; draw $C D$, $C E$, tangents to the arcs $A C$, $C B$, and $O A D$, $O B E$, secants; join $D E$. Since $C D$, $C E$, are drawn in the planes $A O C$, $B O C$, perpendicular to the common intersection $O C$, the angle $D C E$ is the measure of the spherical angle C. By (49.) we have, $OE^2 + OD^2 - 2\ OE \,.\, OD \,.\, \cos DOE = DE^2 = CE^2 + CD^2 - 2\ CE \,.\, CD \,.\, \cos DCE$. Call $C B, a$; $A C, b$; $A B, c$; and observe that $A B$ is the measure of the angle $D O E$; then,

$$\sec^2 a + \sec^2 b - 2 \sec a \,.\, \sec b \,.\, \cos c$$
$$= \tan^2 a + \tan^2 b - 2 \tan a \,.\, \tan b \,.\, \cos C;$$

but $\sec^2 = 1 + \tan^2$, $\sec = \dfrac{1}{\cos}$, and $\tan = \dfrac{\sin}{\cos}$;

therefore,

$$1+\tan^2 a+1+\tan^2 b-\frac{2\cos c}{\cos a\,.\,\cos b}$$

$$=\tan^2 a+\tan^2 b-\frac{2\sin a\,.\,\sin b\,.\,\cos C}{\cos a\,.\,\cos b};$$

hence, $\frac{2\sin a\,.\,\sin b}{\cos a\,.\,\cos b}\,.\,\cos C=\frac{2\cos c}{\cos a\,.\,\cos b}-2$, or

$$2\sin a\,.\,\sin b\,.\,\cos C=2\cos c-2\cos a\,.\,\cos b;$$

from which we obtain,

$$\cos C=\frac{\cos c-\cos a\,.\,\cos b}{\sin a\,.\,\sin b}.$$

This formula is inconvenient for logarithmic computation.

(89.) Since $\cos C=\frac{\cos c-\cos a\,.\,\cos b}{\sin a\,.\,\sin b}$, we have

$$1+\cos C\text{, or }2\cos^2\frac{C}{2}=1+\frac{\cos c-\cos a\,.\,\cos b}{\sin a\,.\,\sin b}$$

$$=\frac{\cos c-(\cos a\,.\,\cos b-\sin a\,.\,\sin b)}{\sin a\,.\,\sin b}$$

$$\frac{\cos c-\cos\overline{a+b}}{\sin a\,.\,\sin b}=\frac{2\sin\frac{a+b+c}{2}\,.\,\sin\frac{a+b-c}{2}}{\sin a\,.\,\sin b}\quad(22).$$

Let $\frac{a+b+c}{2}=S$, then $\frac{a+b-c}{2}=S-c$; hence,

$$\cos^2\frac{C}{2}=\frac{\sin S\,.\,\sin\overline{S-c}}{\sin a\,.\,\sin b}\;\ldots\ldots\;(1.)$$

Again, $1-\cos C$, or $2\sin^2\frac{C}{2}$

$$=1-\frac{\cos c-\cos a\,.\,\cos b}{\sin a\,.\,\sin b}$$

$$=\frac{(\sin a\,.\,\sin b+\cos a\,.\,\cos b)-\cos c}{\sin a\,.\,\sin b}$$

$$=\frac{\cos\overline{a-b}-\cos c}{\sin a\,.\,\sin b}=\frac{2\sin\frac{a+c-b}{2}\,.\,\sin\frac{b+c-a}{2}}{\sin a\,.\,\sin b}$$

$$=\frac{2\sin\overline{S-b}\,.\,\sin\overline{S-a}}{\sin a\,.\,\sin b};$$

hence,

$$\sin^2\frac{C}{2}=\frac{\sin\overline{S-b}\,.\,\sin\overline{S-a}}{\sin a\,.\,\sin b}\;\ldots\ldots\;(2.)$$

Dividing this by the former, we have,

$$\tan^2\frac{C}{2}=\frac{\sin\overline{S-b}\,.\,\sin\overline{S-a}}{\sin S\,.\,\sin\overline{S-c}}\;\ldots\ldots\;(3.)$$

Multiplying the product of the same equations by 4, we obtain $4\sin^2\frac{C}{2}\,.\,\cos^2\frac{C}{2}$, or (23.),

$$\sin^2 C = \frac{4 \sin S \,.\, \sin \overline{S-a} \,.\, \sin \overline{S-b} \,.\, \sin \overline{S-c}}{\sin^2 a \,.\, \sin^2 b} \ldots (4.)$$

All these forms for the determination of an angle when the three sides are given, are convenient for computation by logarithms.

When only one angle is required, the equation (2.) will generally be most appropriate; and when two or all the angles are to be found, the equation (3.) will be the most expeditious, since the same numbers are used in the three calculations (c).

(90.) Suppose two sides and the contained angle are given, to find the other angles and the third side.

We have, by last art.

$$\tan \frac{A}{2} = \sqrt{\frac{\sin \overline{S-b} \,.\, \sin \overline{S-c}}{\sin S \,.\, \sin \overline{S-a}}},$$

$$\text{and } \tan \frac{B}{2} = \sqrt{\frac{\sin \overline{S-a} \,.\, \sin \overline{S-c}}{\sin S \,.\, \sin \overline{S-b}}};$$

hence,

$$\tan \frac{A+B}{2} = \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \,.\, \tan \frac{B}{2}}$$

$$=\frac{\sqrt{\frac{\sin\overline{S-b}\,.\,\sin\overline{S-c}}{\sin S\ \sin\overline{S-a}}}+\sqrt{\frac{\sin\overline{S-a}\,.\,\sin\overline{S-c}}{\sin S\,.\,\sin\overline{S-b}}}}{1-\sqrt{\left(\frac{\sin\overline{S-b}\,.\,\sin\overline{S-c}}{\sin S\,.\,\sin\overline{S-a}}\,.\,\frac{\sin\overline{S-a}\,.\,\sin\overline{S-c}}{\sin S\,.\,\sin\overline{S-b}}\right)}}$$

$$=\frac{\sqrt{\frac{\sin\overline{S-c}}{\sin S}}\left(\sqrt{\frac{\sin\overline{S-b}}{\sin\overline{S-a}}}+\sqrt{\frac{\sin\overline{S-a}}{\sin\overline{S-b}}}\right)}{1-\frac{\sin\overline{S-c}}{\sin S}}$$

$$=\frac{\sqrt{\sin S\,.\,\sin\overline{S-c}}}{\sin S-\sin\overline{S-c}}\left(\sqrt{\frac{\sin\overline{S-b}}{\sin\overline{S-a}}}+\sqrt{\frac{\sin\overline{S-a}}{\sin\overline{S-b}}}\right)$$

$$=\frac{\sqrt{\sin S\,.\,\sin\overline{S-c}}}{\sin S-\sin\overline{S-c}}\left(\frac{\sin\overline{S-b}+\sin\overline{S-a}}{\sqrt{\sin\overline{S-a}\,.\,\sin\overline{S-b}}}\right)$$

$$=\sqrt{\frac{\sin S\,.\,\sin\overline{S-c}}{\sin\overline{S-a}\,.\,\sin\overline{S-b}}}\left(\frac{\sin\overline{S-b}+\sin\overline{S-a}}{\sin S-\sin\overline{S-c}}\right)$$

$$=\cot\frac{C}{2}\,.\,\frac{2\sin\frac{c}{2}\,.\,\cos\frac{a-b}{2}}{2\cos\frac{a+b}{2}\,.\,\sin\frac{c}{2}}\quad(89.)\text{ and }(22.)$$

$$=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}\ \ldots\ldots\ (1.)$$

$$\text{Also } \tan\frac{A-B}{2}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{1+\tan\frac{A}{2}\,.\,\tan\frac{B}{2}}\ \text{(similarly)}$$

$$=\cot\frac{C}{2}\,.\,\frac{\sin\overline{S-b}-\sin\overline{S-a}}{\sin S+\sin\overline{S-c}}$$

$$=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}\ \ldots\ldots\ (2.)$$

Hence, $\frac{A+B}{2}$ and $\frac{A-B}{2}$ are known; their sum gives the value of A, and their difference that of B.

The third side c, is determined from (85).

(91.) The third side, however, may be found without knowing the two angles; that is directly from the data, and this may be done in the following manner.

By (88.), we have,

$$\cos c=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\cos C;$$

but, $\cos C=2\cos^2\frac{C}{2}-1$;

hence,

$$\cos c=\cos a\,.\,\cos b+2\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}$$
$$-\sin a\,.\,\sin b,$$

$$=\cos \overline{a+b}+2 \sin a \,.\, \sin b \,.\, \cos^2 \frac{C}{2};$$

therefore, 1 — cos c, or,

$$2 \sin^2 \frac{c}{2} = 1-\cos \overline{a+b}-2 \sin a \,.\, \sin b \,.\, \cos^2 \frac{C}{2}$$

$$=2 \sin^2 \frac{a+b}{2} -2 \sin a \,.\, \sin b \,.\, \cos^2 \frac{C}{2};$$

hence, we have,

$$\sin^2 \frac{c}{2} = \sin^2 \frac{a+b}{2} - \sin a \,.\, \sin b \,.\, \cos^2 \frac{C}{2}:$$

let $\sin a \,.\, \sin b \,.\, \cos^2 \frac{C}{2} = \sin^2 \theta,$

then,

$$\sin^2 \frac{c}{2} = \sin^2 \frac{a+b}{2} - \sin^2 \theta$$

$$= \sin \left(\frac{a+b}{2} + \theta\right) . \sin \left(\frac{a+b}{2} - \theta\right) \quad (25.)$$

(92.) Otherwise thus:

We have already,

$$\cos c = \sin a \,.\, \sin b \,.\, \cos C + \cos a \,.\, \cos b;$$

let the second side be put under the form,

$$\cos b \,(\sin a \,.\, \tan b \cos C + \cos a),$$

and assume,

$$\tan b \,.\, \cos C = \tan \varphi\,;$$

then we have,

$$\begin{aligned}\cos c &= \cos b\,(\sin a \,.\, \tan \varphi + \cos a)\\ &= \frac{\cos b}{\cos \varphi}\,(\sin a \,.\, \sin \varphi + \cos a \,.\, \cos \varphi)\\ &= \frac{\cos b}{\cos \varphi}\,\cos\,(a-\varphi).\end{aligned}$$

(93.) Another method, for this purpose, may be deduced from the same fundamental theorem.

Since,

$$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C,$$

let the second side of the equation be put under the form,

$$\cos a \,.\, \cos b \left(1 + \frac{\sin a \,.\, \sin b}{\cos a \,.\, \cos b} \,.\, \cos C\right);$$

then, we have,

$$\cos c = \cos a \,.\, \cos b\,(1 + \tan a \,.\, \tan b \,.\, \cos C),$$

$$\text{assume } \tan a \,.\, \tan b \,.\, \cos C = \tan^2 \theta,$$

and,

$$\begin{aligned}\cos c = \cos a \,.\, \cos b\,(1 + \tan^2 \theta) &= \cos a \,.\, \cos b \,.\, \sec^2 \theta\\ &= \frac{\cos a \,.\, \cos b}{\cos^2 \theta}.\end{aligned}$$

When tan a . tan b . cos C is negative, this method, however, fails, since tan θ then becomes impossible.

(94.) Given two angles and the included side to find the other sides and the third angle.

Take the polar triangle (81.) and let a', b', c', be the sides and A' B' C' the angles opposite: then since the sides and angles of the polar triangle are the supplements of the angles and sides of the primitive triangle, we have,

$$A'=\pi-a,\ B'=\pi-b,\ C'=\pi-c, \text{ hence,}$$

$$\frac{A'+B'}{2}=\pi-\frac{a+b}{2};\ \frac{a'+b'}{2}=\pi-\frac{A+B}{2};$$

$$\frac{a'-b'}{2}=\frac{B-A}{2}.$$

Now, by (90......(1.)) we have,

$$\tan\frac{A'+B'}{2}=\frac{\cos\frac{a'-b'}{2}}{\cos\frac{a'+b'}{2}}\cot\frac{C'}{2};\ \text{hence,}$$

$$-\tan\frac{a+b}{2}=\frac{\cos\frac{B-A}{2}}{-\cos\frac{A+B}{2}}\tan\frac{c}{2},\ \text{or}$$

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}\ \ldots\ldots(1.)$$

Again (90......(2.)),

$$\tan\frac{A'-B'}{2}=\frac{\sin\frac{a'-b'}{2}}{\sin\frac{a'+b'}{2}}\cot\frac{C'}{2}$$

hence,

$$-\tan\frac{a-b}{2}=\frac{\sin\frac{B-A}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2},\ \text{or,}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}\ldots\ldots(2.)$$

Hence, $\frac{a+b}{2}$, and $\frac{a-b}{2}$ being known, their sum is the value of a, and their difference that of b.

The third angle C may be found by (85.), or independently of the sides a and b, by a process similar to (91.) or (92.)

(95.) Given the two sides (a, b) and an angle (A) opposite one of them to find the rest.

By (85.) $\sin a : \sin b :: \sin A : \sin B$; from this the angle B will be known.

Also (90.......(1.)), we have,

$$\cot \frac{C}{2} = \frac{\cos \frac{a+b}{2}}{\cos \frac{a-b}{2}} \tan \frac{A+B}{2}$$

hence C is known.

The third side c is found from (85.)

This is one of the ambiguous cases in Spherical Trigonometry, when b is greater than a; for the angle B being found from its sine, has manifestly two values, B and $\pi - B$, and there is nothing in the data to decide whether the triangle $A\,C\,B$, or $A\,C\,B'$, (fig. 25.) in which $C\,B = C\,B'$, is to be taken.

(96.) Given the two angles (A, B) and the side (a) opposite one of the angles, to determine the other parts.

By (85.), we have $\sin A : \sin B :: \sin a : \sin b$; this will determine b.

From (94......(1.)),

$$\tan \frac{c}{2} = \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}} \tan \frac{a+b}{2}$$

hence c will be known.

The third angle C is found from (85.)

In this case, when B is greater than A, either of the triangles $A\,B\,C$, $D\,B\,C$ (fig. 25.) in which $D\,C$ produced makes $C\,B' = A\,C$, satisfies the given conditions.

These are the only ambiguous cases of oblique spherical triangles.

(97.) If the three angles are given to find the three sides, we may take the polar triangle in which the three sides will be known, and find by (89.), the three angles, the supplements of which will be the sides required. This is, however, a case which never occurs in any applications of Spherical Trigonometry*.

(98.) In the solution of quadrantal triangles, if we take the corresponding polar triangle, it will evidently be right angled. Hence Napier's rules may be applied to quadrantal triangles, if we take for the circular parts the complements of the sides, the complement of the angle opposite the quadrant, and the two angles. But as there is some difficulty in the determination of the signs, it will be more con-

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* Professor Airey.

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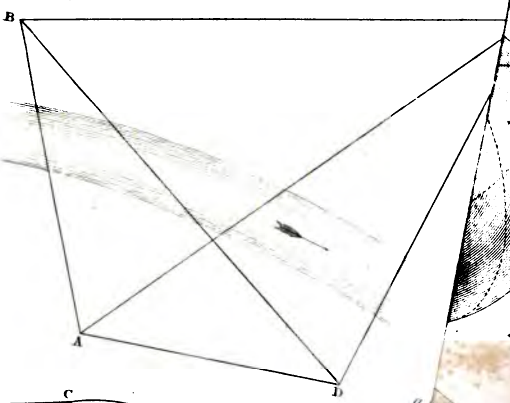

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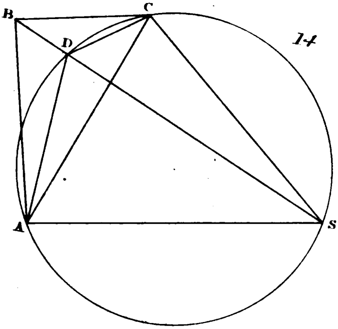

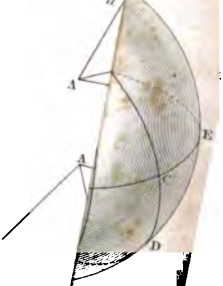

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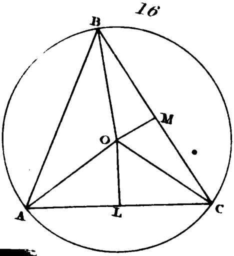

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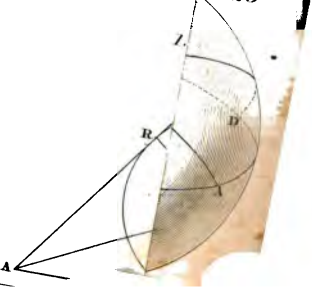

venient to make use of the general formulæ in (85.) and (88.), which, when applied to quadrantal triangles, are always much simplified.

(99.) We have now detailed all the formulæ necessary for the solution of spherical triangles. Some of the cases may be solved by drawing a perpendicular from one of the angles to the opposite side, and applying Napier's rules for the solution of right angled triangles. But as some difficulty may arise to a young student, in deciding whether the perpendicular falls within or without the triangle, it will, perhaps, be found to be more expeditious, to use the preceding general formulæ (*c.*)

(100.) We shall now proceed to the practical solution of spherical triangles, in every case of which it will be necessary to bear in mind that every side and angle must be less than 180°, and that the greater angles are opposite to the greater sides.

Example I. In the spherical triangle, ABC, suppose there were given the side $a=104^\circ\ 23'$, the side $b=67^\circ\ 3'$, and the side $c=52^\circ\ 42'$, and that it was required to find the angle A, we should obtain it by the following operation.

$$\begin{array}{rrl} 104^\circ & 23' & =a \\ 67^\circ & 3 & =b \\ 52^\circ & 42 & =c \\ \hline 224^\circ & 8 & \\ \hline 112^\circ & 4 & =S. \end{array}$$

$\therefore\ \overline{S-b}=45^\circ\ 1' \qquad \overline{S-c}=59^\circ\ 22'.$

By (89...(2), $\sin\frac{A}{2}=\sqrt{\frac{\sin\overline{S-b}\,.\,\sin\overline{S-c}}{\sin b\,.\,\sin c}}.$

sin 67° 3′, co ar	0·0358132
sin 52° 42′..................	0·0993743
sin 45° 1′ log.............	9·8496113
sin 59° 22′................	9·9347235
	19·9195223
$\sin\frac{A}{2}$	9·9597611

This corresponds to $\frac{A}{2}=65^\circ\ 42'\ 53''$

$\therefore\ A=131^\circ\ 25'\ 46''.$

Example II. Given the side $a=118° 20'$, the side $b=38° 9'$, and the included angle $C=40°$, to determine the angles A and B and the side c.

$$\text{By (90.)} \tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}$$

$$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2},$$

in which,

$$\frac{a-b}{2}=40° 5' 30'', \quad \frac{a+b}{2}=78° 14' 30'', \quad \frac{C}{2}=20°;$$

hence we shall obtain the angles A and B by the following computation:

cos 78° 14′ 30″, co ar...	0·6908296
cos 40° 5′ 30″, log.......	9·8836700
cot 20°	10·4389341
$\tan\frac{A+B}{2}$	11·0134337

$$\therefore \frac{A+B}{2}=84° 27' 44''$$

sin 78° 14′ 30″, co ar.	0·0092103
sin 40° 5′ 30″, log,	9·8088942
cot 20°	10·4389341
tan $\frac{A-B}{2}$	10·2570386

$$\therefore \frac{A-B}{2} = 61° \, 2' \, 39''.$$

$$\therefore A = 145° \, 30' \, 23'',$$

$$B = 23° \, 25' \, 5''.$$

Now to find the third side c, we have the proportion,

$$\sin B : \sin C :: \sin b : \sin c.$$

sin 40°, log	9·8080675
sin 38° 9′	9·7907933
	19·5988608
sin 23° 25′ 5″	9·5992684
sin c	9·9995924

hence, $c = 87° \, 31' \, 5''$.

Example III. Suppose we have given the angle $A=112° 25'$, the angle $B=73° 15'$, and the contained side $c=24° 38'$ to find the rest.

$$\text{By (94.) } \tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}\text{ ...... (1.)}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}\text{ ...... (2.)}$$

$$\frac{A-B}{2}=19° 35' \qquad \frac{A+B}{2}=92° 50' \qquad \frac{c}{2}=12° 19' :$$

hence we shall obtain the values of the sides a and b by computing, as follows:

cos 19° 35′, log	9·9741224
tan 12° 19′................	9·3391333
	19·3132557
cos 92° 50′	8·6939980
$\tan\frac{a+b}{2}$	10·6192577

The number corresponding to this logarithm in the tables, is 76° 29′ 18″; but since $\cos\frac{A+B}{2}$ is negative, therefore (equ. 1.) $\tan\frac{a+b}{2}$ is also negative; hence we must take the supplement of the angle in the tables for the value of $\frac{a+b}{2}$. We have therefore, $\frac{a+b}{2}=103^\circ\ 30'\ 42''$.

$$\begin{array}{ll}
\sin 19^\circ\ 35', \log \ldots\ldots\ldots\ldots & 9\cdot5252749 \\
\tan 12^\circ\ 19' \ldots\ldots\ldots\ldots\ldots\ldots & 9\cdot3391333 \\
\cline{2-2}
 & 18\cdot8644082 \\
\sin 92^\circ\ 50' \ldots\ldots\ldots\ldots\ldots\ldots & 9\cdot9994688 \\
\cline{2-2}
\tan \frac{a-b}{2} \ldots\ldots\ldots\ldots\ldots\ldots & 8\cdot8649394
\end{array}$$

$$\therefore \frac{a-b}{2}=4^\circ\ 11'\ 26''\cdot5.$$

$$\therefore a = 107^\circ\ 42'\ 8''\cdot5,$$

$$b = 99^\circ\ 19'\ 16''\cdot5.$$

To find the angle C, we have,

$$\sin b : \sin c :: \sin B : \sin C.$$

$$\begin{array}{ll}
\sin 24^\circ\ 38', \log \ldots\ldots\ldots\ldots & 9\cdot6199378 \\
\sin 73^\circ\ 15' \ldots\ldots\ldots\ldots\ldots\ldots & 9\cdot9811711 \\
\cline{2-2}
 & 19\cdot6011089 \\
\sin 99^\circ\ 19'\ 16''\cdot5 \ldots\ldots\ldots & 9.9942272 \\
\cline{2-2}
\sin C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 9\cdot6068817
\end{array}$$

$$\therefore C=23^\circ\ 51'\ 27''\cdot5.$$

Example IV. Let there be given the side $a=75^\circ\,33'$, the side $b=18^\circ\,5'$, and the angle $A=130^\circ$ to find the angles B and C and the third side.

To find the angle B, we have,

$$\sin a : \sin b :: \sin A : \sin B.$$

sin 18° 5′, log…………	9·4919216
sin 130° ………………	9·8842540
	19·3761756
sin 75° 33′ ……………	9·9860394
sin B… ………………	9·3901362

Now, there are two angles answering to this logarithmic sine, viz: 14° 12′ 51″, and its supplement; but the latter being greater than A, is inadmissible; we must therefore restrict our choice to the former.

Hence, $B=14^\circ\,12'\,51''$.

To find the angle C.

$$\text{By (95.)}\ \cot\frac{C}{2}=\frac{\cos\frac{a+b}{2}}{\cos\frac{a-b}{2}}\tan\frac{A+B}{2}$$

in which,

$$\frac{a+b}{2}=46^\circ\,49' \quad \frac{a-b}{2}=28^\circ\,44' \quad \frac{A+B}{2}=72^\circ\,6'\,25''\tfrac{1}{2}$$

cos $46^\circ\ 49'$, log	$9 \cdot 8352688$
tan $72^\circ\ 6'\ 25''\frac{1}{2}$.........	$10 \cdot 4909929$
	$20 \cdot 3262617$
cos $28^\circ\ 44'$................	$9 \cdot 9429335$
cot $\frac{C}{2}$	$10 \cdot 3833232$

$$\therefore \frac{C}{2} = 22^\circ\ 28'\ 27'' \cdot 3$$

$$C = 44^\circ\ 56'\ 54'' \cdot 6.$$

Lastly, to determine the third side c, we have,

$$\sin A : \sin C :: \sin a : \sin c.$$

sin $44^\circ\ 56'\ 54'' \cdot 6$, log..	$9 \cdot 8490943$
sin $75^\circ\ 33'$................	$9 \cdot 9860394$
	$19 \cdot 8351337$
sin 130°..................	$9 \cdot 8842540$
sin c........................	$9 \cdot 9508797$

$$\therefore c = 63^\circ\ 15'\ 36''.$$

Example V. Suppose $A=130^\circ$, $B=14^\circ\ 12'\ 51''$, and $a=75^\circ\ 33'$, required b, c and C.

To find b.

$$\sin A : \sin B :: a : b.$$

sin 14° 12′ 51″, log	9·3901362
sin 75° 33′................	9·9860394
	19·3761756
sin 130°...................	9·8842540
sin b	9·4919216

Now, since b must be less than a, there exists no ambiguity; therefore,

$$b=18^\circ\ 5'.$$

To find the side c, we have,

$$\tan\frac{c}{2}=\frac{\cos\dfrac{A+B}{2}}{\cos\dfrac{A-B}{2}}\tan\frac{a+b}{2}$$

$\frac{A+B}{2}$ being$=72^\circ\ 6'\ 25''\tfrac{1}{2}$, $\frac{A-B}{2}=57^\circ\ 53'\ 34''\tfrac{1}{2}$, $\frac{a+b}{2}=46^\circ\ 49'$.

cos 72° 6′ 25″½, log	9·4874763
tan 46° 49′	10·0275587
	19·5150350
cos 57° 53′ 34″½	9.7255060
tan $\frac{c}{2}$......................	9·7895290

$$\text{hence, } \frac{c}{2} = 31^\circ\, 37'\, 48''$$

$$c = 63^\circ\, 15'\, 36''.$$

To find C.

$$\sin a : \sin c :: \sin A : \sin C.$$

sin 63° 15′ 36″, log.....	9·9508797
sin 130°..................	9·8842540
	19·8351337
sin 75° 33′	9·9860394
sin C.....................	9·8490943

$$\therefore C = 44^\circ\, 56'\, 54''\cdot 6.$$

Example VI. Let it be required to determine the shortest distance between two places upon the earth; suppose between Constantinople and Tobolsk in Asiatic Russia: the latitude and longitude of the former being 41° N. and 29° E. and those of the latter 58° N. and 69° E.

Let P (fig. 26.) represent the north pole of the earth, $P\,G$, $P\,C$, and $P\,T$, the meridians of Greenwich, Constantinople, and Tobolsk, respectively; and conceive C, T, the points representing the two places, to be joined by the arc of a great circle; this line will evidently be the shortest distance between them.

In the spherical triangle, *P C T*, the two sides, *P C*, *P T*, are known, since they are the complements of the latitudes of the two places, and we likewise know the angle *C P T*, at the pole, since it is manifestly the difference in longitude of the two points *C* and *T*.

$$\text{Let } P\,C = 49^\circ = a$$
$$P\,T = 32^\circ = b$$
$$C\,P\,T = 40^\circ = C,$$

and if we call *C T*, *c*, we have, for determining it, the formulæ,

$$\tan \phi = \cos C \,.\, \tan b,$$
$$\cos c = \frac{\cos b \,.\, \cos (a-\phi)}{\cos \phi} \text{ (92).}$$

cos 40°, log .... 9·8842540
tan 32° .... 9·7957892

tan ϕ .... 9·6800432

$$\therefore \phi = 25^\circ\ 34'\ 46''.$$

Hence $(a-\phi) = 23^\circ\ 25'\ 14''$.

cos 32°, log................	9 · 9284205
cos 23° 25′ 14″...........	9 · 9626592
	19 · 8910797
cos 25° 34′ 46″...........	9 · 9552005
cos *c*........................	9 · 9358792

$$\therefore c \text{ or } CT = 30^\circ\, 22'\, 28''.$$

Now, if we take the length of a degree to be 69 · 12 miles, we shall find the distance between Constantinople and Tobolsk to be $30^\circ \cdot 3744 \times 69 \cdot 12 = 2099 \cdot 2$ miles.

Example VII. In the extension of the late French Trigonometrical Survey, the spherical excess of the triangle which connected Spain with Iviza, amounted to 39″; what was the area of the triangle, supposing the earth a sphere, and the radius 3960 miles?

Denoting the three angles of the spherical triangle by A, B, C, we have, by (80.)

$$\text{area of the triangle} = \text{rad}^2 \times (A + B + C - \pi);$$

the quantity within the vinculum representing the spherical excess, expressed in terms of the radius 1.

Hence,

$$\text{area} = 3960^2 \times 39 \times 0 \cdot 000004848137,$$

or,

$$\begin{aligned}\log \text{area} &= 2 \log 3960 + \log 39 + 4 \cdot 68557487^* \\ &= 11 \cdot 88096525 + \log 39 \\ &= 13 \cdot 47202986.\end{aligned}$$

$$\therefore \text{area} = 2965 \cdot 03 \text{ square miles.}$$

Example VIII. In a quadrantal triangle, $A\,B\,C$, suppose we have given the side $b = 38^\circ\ 19'$, the side $c = 90^\circ$, and the angle $C = 115^\circ\ 58'$; to find the other parts.

By (88.) we have the formula

$$\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b};\ \text{but } \cos c = 0.$$

$$\therefore \cos C = -\cot a \,.\, \cot b,$$

or,

$$\cot a = -\frac{\cos C}{\cot b}$$

---

* In calculating the value of the area, we increased the logarithm of the value of $1''$ in terms of radius 1, by 10, and therefore it was necessary to diminish the logarithm of the area by 10. The only reason for employing this process is to obviate the inconvenience of using negative indices.

$$=\frac{\cos(\pi-C)}{\cot b}=\frac{\cos 64^\circ\ 2'}{\cot 38^\circ\ 19'}$$

cos 64° 2′, log	9 · 6413235
cot 38° 19′	10 · 1022493
cot *a*	9 · 5390742

$$\therefore a=70^\circ\ 54'\ 52''.$$

The sides *a* and *b* being known, we determine the opposite angles by the application of (85.), which gives $A=58^\circ\ 10'\ 17''$, and $B=33^\circ\ 52'\ 38''$.

Example IX. The three sides of a spherical triangle are 79° 18′, 55° 30′, and 115° 46′, what are the three angles?

Answer, 54° 24′ 52″, 43° 0′ 23″, and 131° 48′ 39″.

Example X. In a spherical triangle occurring in the Trigonometrical Survey, the sum of the angles amounted to 180° 0′ 1 · 25″, what was the area of the triangle?

Area = 95 · 033 square miles.

Example XI. What is the distance between the observatories of Greenwich and Paris, the latitudes of the two places being 51° 28′ 40″ N. and 48° 50′ 15″ N., and their difference in longitude 2° 20′ 24″?

The arc between the two observatories = 3° 2′ 20″.

Example XII. Suppose the two angles of a spherical triangle to be 58° 5′ 4″, and 62° 34′ 6″, and the included side 122°; required the rest.

Answer, the two sides are 70° and 79° 17′ 14″, and the third angle 130°.

Example XIII. Required the length of a degree of longitude at Greenwich, in latitude 51° 28′ 40″ N., supposing the earth to be a sphere, and the radius =3960 miles. (Vid. fig. 19.)

Answer, 43 · 046 miles.

SECTION VII.

ON THE EXPANSION OF TRIGONOMETRICAL QUANTITIES INTO SERIES, ETC.

(101.) In the preceding part of the work, we have made use only of the elementary theorems in Geometry, and the simpler processes of Algebra; in this section it will be necessary for the student to be somewhat acquainted with the higher parts of Algebra, theory of Equations, Logarithms, &c.

(102.) If we multiply together the two factors,

$$\cos A \pm \sqrt{-1}\sin A \text{ and } \cos B \pm \sqrt{-1}\sin B,$$

the product $=$

$$\cos A \,.\, \cos B - \sin A \,.\, \sin B \pm \sqrt{-1}\,(\sin A \,.\, \cos B + \cos A \,.\, \sin B),$$

which, by (20.), may be reduced to the form,

$$\cos (A+B) \pm \sqrt{-1}\sin (A+B).$$

From this principle, we form successively the following equations:

$$\begin{aligned}
(\cos A \pm \sqrt{-1}\sin A)^2 &= (\cos 2A \pm \sqrt{-1}\sin 2A),\\
(\cos A \pm \sqrt{-1}\sin A)^3 &= (\cos 3A \pm \sqrt{-1}\sin 3A),\\
(\cos A \pm \sqrt{-1}\sin A)^4 &= (\cos 4A \pm \sqrt{-1}\sin 4A),\\
\&c. \quad &= \quad \&c.
\end{aligned}$$

hence, generally,

$$(\cos A + \sqrt{-1}\sin A)^n = \cos nA + \sqrt{-1}\sin nA,$$

$$(\cos A - \sqrt{-1}\sin A)^n = \cos nA - \sqrt{-1}\sin nA.$$

This is Demoivre's theorem.

(103.) Lemma. When the arc is diminished indefinitely the limiting ratio of the arc, sine and tangent is one of equality.

Let $A\,P$ (fig. 3.) be an arc, $P\,M$ its sine, and

$A\,T$ its tangent. Since the sector $A\,O\,P$ is less than the triangle $O\,A\,T$, we have,

$\frac{1}{2}\,A\,O\,.\,A\,P$ less than $\frac{1}{2}\,A\,O\,.\,A\,T$, or $A\,P$ less than $A\,T$.

Also, the arc $A\,P$, being greater than the chord, is therefore greater than the sine. But, $\frac{\tan A\,P}{\sin A\,P} = \frac{1}{\cos A\,P} = 1$ when $A\,P = 0$; hence the arc, sine, and tangent, tend to equality, when the arc is diminished indefinitely.

(104.) To express the sine and cosine in terms of the arc.

Resuming the equations in (102.), we have,

$$\cos n\,A + \sqrt{-1}\,\sin n\,A = (\cos A + \sqrt{-1}\,\sin A)^n,$$

$$\cos n\,A - \sqrt{-1}\,\sin n\,A = (\cos A - \sqrt{-1}\,\sin A)^n;$$

by addition and subtraction,

$$\cos n\,A = \tfrac{1}{2}\,\{(\cos A + \sqrt{-1}\,\sin A)^n + (\cos A - \sqrt{-1}\,\sin A)^n\};$$

$$\sqrt{-1}\,\sin n\,A = \tfrac{1}{2}\,\{(\cos A + \sqrt{-1}\,\sin A)^n - (\cos A - \sqrt{-1}\,\sin A)^n\}.$$

Expanding these powers into series, we shall have,

$$\cos n\,A = \cos^n A - \frac{n\,(n-1)}{1\,.\,2}\cos^{n-2} A\,.\,\sin^2 A$$

$$+\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}\cos^{n-4} A\,.\,\sin^4 A - \&c.$$

$$\sin n\,A = n\cos^{n-1} A\,\sin A - \frac{n\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3}$$

$$.\cos^{n-3} A\,.\,\sin^3 A + \&c.$$

or, since $\cos^{n-1} A = \dfrac{\cos^n A}{\cos A}$, $\cos^{n-2} A = \dfrac{\cos^n A}{\cos^2 A}$, &c.

$$\cos n\,A = \cos^n A\,.\left\{1 - \frac{n\,(n-1)}{1\,.\,2}\tan^2 A\right.$$

$$\left.+\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}\tan^4 A - \&c.\right\};$$

$$\sin n\,A = \cos^n A\left\{n\tan A\right.$$

$$\left.-\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\tan^3 A + \&c.\right\}:$$

let $n\,A = x$ $\therefore$ $n = \dfrac{x}{A}$; hence we have

$$\cos x = \cos^n A\left\{1 - \frac{x\,(x-A)}{1\,.\,2}\,.\,\frac{\tan^2 A}{A^2}\right.$$

$$+\frac{x\,(x-A)\,(x-2\,A)\,(x-3\,A)}{1\,.\,2\,.\,3\,.\,4}\,.\,\frac{\tan^4 A}{A^4}-\text{\&c.}\Big\}$$

$$\sin x=\cos^n A\,\Big\{x\,.\,\frac{\tan A}{A}$$

$$-\frac{x\,(x-A)\,.\,(x-2\,A)}{1\,.\,2\,.\,3}\,.\,\frac{\tan^3 A}{A^3}+\text{\&c.}\Big\}.$$

Now, these series being true for any value of A, are therefore true when $A=0$; in this case, $\cos A=1$ $\therefore$ $\cos^n A=1$; also $\frac{\tan A}{A}=1$:

hence, taking the limits,

$$\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{\&c.}$$

$$\sin x=x-\frac{x^3}{1\,.\,2\,.\,3}+\frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}-\text{\&c.}$$

(105.) Let e be the base of the Naperian system of logarithms, then it is well known that

$$e^x=1+\frac{x}{1}+\frac{x^2}{1\,.\,2}+\frac{x^3}{1\,.\,2\,.\,3}+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}+\text{\&c.}$$

if x be successively changed to $x\sqrt{-1}$ and $-x\sqrt{-1}$, we shall have,

$$e^{x\sqrt{-1}}=1+\frac{x\sqrt{-1}}{1}-\frac{x^2}{1\,.\,2}-\frac{x^3\sqrt{-1}}{1\,.\,2\,.\,3}$$

$$+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}+\text{\&c.}$$

$$e^{-x\sqrt{-1}}=1-\frac{x\sqrt{-1}}{1}-\frac{x^2}{1\,.\,2}+\frac{x^3\sqrt{-1}}{1\,.\,2\,.\,3}$$

$$+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}-\text{\&c.}$$

by addition and subtraction, we obtain,

$$e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}$$

$$=2\left(1-\frac{x^2}{1\,.\,2}+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}-\text{\&c.}\right)=2\cos x\,.\,.\,(1.)$$

$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}$$

$$=2\sqrt{-1}\left(x-\frac{x^3}{1\,.\,2\,.\,3}+\frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}-\text{\&c.}\right)$$

$$=2\sqrt{-1}\sin x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.)$$

hence,

$$\cos x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}\ \ldots\ldots\ldots\ldots\ (3.),$$

$$\sin x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}\ \ldots\ldots\ldots\ldots\ (4.)$$

These two formulæ are due to Euler.

(106.) By adding and subtracting the equations, (1.), (2.), we have,

$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$, and

$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x$;

hence, by (102.),

$$e^{nx\sqrt{-1}} = \cos nx + \sqrt{-1}\sin nx,$$

$$\text{and } e^{-nx\sqrt{-1}} = \cos nx - \sqrt{-1}\sin nx.$$

Also,

$$e^{(x+y+\ldots\ldots)\sqrt{-1}} = \cos(x+y+\ldots\ldots)$$

$+\sqrt{-1}\sin(x+y+\ldots\ldots)$, and

$$e^{(-x-y-\ldots\ldots)\sqrt{-1}} = \cos(x+y+\ldots\ldots)$$

$-\sqrt{-1}\sin(x+y+\ldots\ldots)$,

whatever be the number of arcs $x, y\ldots\ldots$

By dividing equation (4.) by (3.), we have,

$\tan x = \frac{1}{\sqrt{-1}} \cdot \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}$, or (by multiplying the numerator and denominator by $e^{x\sqrt{-1}}$)

$$= \frac{1}{\sqrt{-1}} \cdot \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}.$$

(107.) Dividing the second equation in (104.) by the first, we have,

$$\tan x = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.}$$ or, (by actual division),

$$= x + \frac{2\,x^3}{1.2.3} + \frac{16\,x^5}{1.2.3.4.5} + \&c.$$

(108.)

Since $e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\cos x$,

let $e^{x\sqrt{-1}} = y$, then,

$$y + \frac{1}{y} = 2\cos x,$$

the solution of this quadratic gives,

$$y = \cos x + \sqrt{-1}\sin x,$$

$\therefore \frac{1}{y} = \frac{1}{\cos x + \sqrt{-1}\sin x}$, or, (by multiplying the numerator and denominator by $\cos x - \sqrt{-1}\sin x$),

$$= \frac{\cos x - \sqrt{-1}\sin x}{\cos^2 x + \sin^2 x} = \cos x - \sqrt{-1}\sin x.$$

Therefore,

$$y^n + \frac{1}{y^n}$$

$$= (\cos x + \sqrt{-1} \sin x)^n + (\cos x - \sqrt{-1} \sin x)^n$$
$$= \cos n x + \sqrt{-1} \sin n x + \cos n x - \sqrt{-1} \sin n x$$
$$= 2 \cos n x.$$

Similarly, if

$$y - \frac{1}{y} = 2\sqrt{-1} \sin x;$$

then will $y^n - \frac{1}{y^n} = 2\sqrt{-1} \sin n x.$

(109.) To express the arc in terms of the tangent.

Since, $e^{x\sqrt{-1}} = \cos x + \sqrt{-1} \sin x$
$$= \cos x (1 + \sqrt{-1} \tan x),$$

taking the Naperian logarithms,

$$x\sqrt{-1} = \log \cos x + \log (1 + \sqrt{-1} \tan x),$$

now, $\log (1 + x)$ expressed in a series,

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

$$\therefore x\sqrt{-1}=\log\cos x+\sqrt{-1}\tan x$$

$$+\frac{\tan^2 x}{2}-\sqrt{-1}\,.\,\frac{\tan^3 x}{3}-\frac{\tan^4 x}{4}+\&c.$$

equating the impossible parts on each side, and we have,

$$x=\tan x-\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}-\frac{\tan^7 x}{7}+\&c.$$

(110.) To find the value of π.

Let $\tan x=\frac{1}{5}$,

then,

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}=\frac{5}{12},$$

and,

$$\tan 4x=\frac{2\tan 2x}{1-\tan^2 2x}=\frac{\frac{5}{6}}{1-\frac{25}{144}}=\frac{120}{119};$$

$$\therefore 4x \text{ is greater than } \frac{\pi}{4}.$$

Assume $y=4x-\frac{\pi}{4}$, then (since $\tan\frac{\pi}{4}=1$),

$$\tan y = \frac{\tan 4\,x - 1}{1 + \tan 4\,x} = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{1}{239}:$$

$$\therefore \frac{\pi}{4} = 4\,x - y = 4 \text{ arc, } \tan = \frac{1}{5} - \text{arc, } \tan = \frac{1}{239},$$

or, as it is now expressed, from a notation in the higher mathematics,

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}.$$

Now, (109.)

$$4 \tan^{-1} \frac{1}{5} = 4 \left(\frac{1}{5} - \frac{1}{3} \cdot \frac{1}{5^3} + \frac{1}{5} \cdot \frac{1}{5^5} - \frac{1}{7} \cdot \frac{1}{5^7} + \&c.\right)$$

and,

$$\tan^{-1} \frac{1}{239} = \frac{1}{239} - \frac{1}{3} \cdot \frac{1}{239^3} + \frac{1}{5} \cdot \frac{1}{239^5} - \&c.;$$

$$\therefore \frac{\pi}{4} = 4 \left(\frac{1}{5} - \frac{1}{3} \cdot \frac{1}{5^3} + \frac{1}{5} \cdot \frac{1}{5^5} - \frac{1}{7} \cdot \frac{1}{5^7} + \&c.\right)$$

$$- \left(\frac{1}{239} - \frac{1}{3} \cdot \frac{1}{239^3} + \frac{1}{5} \cdot \frac{1}{239^5} - \&c.\right)$$

These series are so convergent, that by taking nine terms of the first and four of the second, the value of π will be obtained correct to 12 places of decimals; and by taking eighteen of the first and six of the latter, its value will be true to upwards of twenty places.

(111.) This is the celebrated problem called the *squaring of the circle*, which occasionally occupied the attention of mathematicians from the time of Archimedes to a comparatively late date. Archimedes shewed that the value of the semicircumference to radius 1, or π, was greater than $3\frac{10}{71}$, and less than $3\frac{10}{70}$. Ludolph Van Cuelen, a Dutchman, by a very laborious method, carried the approximation to thirty-six decimals. Adrian Metius, a cotemporary with Van Cuelen, gave a very remarkable value, $\frac{355}{113}$, being composed of the first three odd numbers; this is correct to the seventh decimal. Abraham Sharp, an indefatigable calculator at Greenwich observatory, extended the approximation, by a method of Dr. Halley's, to seventy-two decimals. John Machin, to whom we are indebted for the formula

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{239},$$

carried forward the value of π to one hundred places; and some have even had the patience to extend it to the hundred and twenty-seventh, or even to the hundred and fortieth place [*].

"Accordingly, this problem, which engaged the attention of geometers so deeply when their methods of approximation were less perfect, is now degraded to the rank of those idle questions, with which no one possessing the slightest tincture of geometrical science will occupy any portion of his time."—*Legendre.*

SECTION VIII.

(112.) To find the area of the circle.

Let r represent the radius and c the circumference, then,

$$\text{area} = \frac{r \cdot c}{2} = r^2 . \pi = r^2 \times 3 \cdot 141593$$

$$\text{or} = \frac{c^2}{4\pi} = c^2 \times 0 \cdot 079580.$$

---

[*] The following is Lagni's approximation:

$\pi = 3{\cdot}14159$ 26535 89793 23846 26433 83279 50288 41971
69399 37510 58209 74944 59230 78164 06286 20899
86280 34825 34211 70679 82148 08651 32723 06647
09384 46+. (Mem. Acad. of Sciences, 1719.)

(113.) To find the number of degrees, &c., in an arc which is equal to the radius.

Since angles are proportional to the arcs on which they stand,

$3 \cdot 14159\ldots : 1 :: 180^\circ : x$, the number required;

$$\therefore x = \frac{180^\circ}{3 \cdot 14159\ldots} = 57^\circ \cdot 2957795$$
$$= 57^\circ\, 17'\, 44''\, 48'''.$$

(114.) To find the roots of a quadratic equation.

1°. Let the equation be $x^2 + p\,x - q = 0$;

$$\therefore x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$
$$= -\frac{p}{2}\left(1 \pm \sqrt{1 + \frac{4q}{p^2}}\right).$$

Assume, $\dfrac{4q}{p^2} = \tan^2\theta$; then if the sign + be taken,

$$x = -\frac{p}{2}(1 + \sec\theta)$$
$$= -\frac{p}{2}\left(1 + \frac{1}{\cos\theta}\right)$$
$$= -\frac{p}{2}\left(\frac{\cos\theta + 1}{\cos\theta}\right)$$
$$= -\frac{p\cos^2\dfrac{\theta}{2}}{\cos\theta}.$$

Similarly, if the sign — be taken,

$$x = p \cdot \frac{\sin^2 \frac{\theta}{2}}{\cos \theta}.$$

These are the two roots of the equation $x^2 + p\,x + q = 0$.

2°. Let the equation be $x^2 - p\,x + q = 0$,

$$\therefore x = \frac{p}{2}\left(1 \pm \sqrt{1 - \frac{4q}{p^2}}\right).$$

Let $\frac{4q}{p^2} = \sin^2\theta \therefore x = \frac{p}{2}(1 \pm \cos\theta)$.

Hence, $x = p \cos^2 \frac{\theta}{2}$, and $= p \sin^2 \frac{\theta}{2}$, which are the two roots of the equation $x^2 - p\,x + q = 0$.

3°. If the equation be $x^2 - p\,x - q = 0$, the roots will evidently be the roots of the first equation with their signs changed.

4°. Let the equation be $x^2 + p\,x + q = 0$, the roots will then be those of the second equation with their signs changed.

It will readily be seen that this method of computing the roots of a quadratic equation can only be used with advantage when the numbers, p and q, consist of several figures.

(115.) To find the roots of a cubic equation.

Let the proposed cubic be deprived of its second term, so as to be reduced to the form

$$x^3 - m\,x - n = 0.$$

Let $x = p \cos \theta$, then,

$$\cos \theta = \frac{x}{p}, \text{ and } x^3 = p^3 \cos^3 \theta.$$

Now, by (27.),

$$\cos 3\,\theta = 4 \cos^3 \theta - 3 \cos \theta, \text{ or,}$$

$$4\,\frac{x^5}{p^3} - 3\,\frac{x}{p} = \cos 3\,\theta,$$

$$\therefore x^3 - \frac{3}{4}p^2\,x - \frac{p^3}{4}\cos 3\,\theta = 0;$$

making this coincide with the proposed equation, we shall have,

$$\frac{3}{4}p^2 = m, \frac{p^3}{4}\cos 3\,\theta = n,$$

$$\therefore p = 2\sqrt{\frac{m}{3}}, \cos 3\,\theta = \frac{4\,n}{p^3} = \frac{4\,n}{8\left(\frac{m}{3}\right)^{\frac{3}{2}}} = \frac{n}{2}\sqrt{\frac{27}{m^3}}.$$

Hence, if we determine $\cos 3\,\theta$ by the equation,

$$\log\cos 3\theta = \log\frac{n}{2} + \tfrac{1}{2}(\log 27 - 3\log m) + 10 \ldots (1.)$$

and trisect it, we shall have θ:

then to find x, we have the equation,

$$x = p\cos\theta = 2\sqrt{\frac{m}{3}}\,.\cos\theta, \text{ or,}$$

$$\log x = \log 2 + \tfrac{1}{2}\log\frac{m}{3} + \log\cos\theta - 10 \ldots\ldots (2.)$$

This is one of the roots of the cubic equation.

But there are two other arcs having the same cosine as 3θ, viz $2\pi + 3\theta$ and $2\pi - 3\theta$, which will satisfy the equation (1.), therefore the other roots are

$$2\sqrt{\frac{m}{3}}\,.\cos\frac{2\pi + 3\theta}{3}, \text{ and } 2\sqrt{\frac{m}{3}}\,.\cos\frac{2\pi - 3\theta}{3}.$$

Since the cosine of an arc is always less than 1, it is necessary, in order that this method may succeed, that

$$\cos 3\theta, \text{ or } \frac{n}{2}\sqrt{\frac{27}{m^3}} \text{ should be less than 1.}$$

$$\therefore \frac{n^2}{4}\,.\,\frac{27}{m^3} \text{ is less than 1,}$$

$$\text{or, } \frac{n^2}{4} \text{ less than } \frac{m^3}{27}.$$

Hence this method determines the roots of a cubic equation only when Cardan's fails; that is, when all the roots are possible.

SECTION IX.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

(116.) To express the sine and cosine of one minute in terms of radius 1.

From (23.) we have the formulæ,

$$\cos A = \sqrt{\frac{1+\cos 2A}{2}};$$

let $A = 30°$, then

$$\cos 30° = \frac{\sqrt{3}}{2} = 0{\cdot}8660254038 = C \text{ (suppose)}$$

$$\cos\frac{30°}{2} = \sqrt{\frac{1+C}{2}} = 0{\cdot}9659258263 = C_1$$

$$\cos\frac{30°}{2^2} = \sqrt{\frac{1+C_1}{2}} = 0{\cdot}9914448614 = C_2$$

$$\cos\frac{30°}{2^3} = \sqrt{\frac{1+C_2}{2}} = 0{\cdot}9978589232 = C_3$$

$$\cos\frac{30^\circ}{2^4} \quad \cdots\cdots\cdots \quad = 0{\cdot}9994645875$$

$$\cos\frac{30^\circ}{2^5} \quad \cdots\cdots\cdots \quad = 0{\cdot}9998661399$$

$$\cos\frac{30^\circ}{2^6} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999665346$$

$$\cos\frac{30^\circ}{2^7} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999916336$$

$$\cos\frac{30^\circ}{2^8} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999979084$$

$$\cos\frac{30^\circ}{2^9} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999994775$$

$$\cos\frac{30^\circ}{2^{10}} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999998693 = C_{10}$$

$$\cos\frac{30^\circ}{2^{11}} \quad \cdots\cdots\cdots \quad = 0{\cdot}9999999673 = C_{11}$$

Now, from the formula $\sin A = \sqrt{\dfrac{1-\cos 2A}{2}}$,

we shall have,

$$\sin\frac{30^\circ}{2^{11}} = \sqrt{\frac{1-C_{10}}{2}} = 0{\cdot}00025566$$

$$\sin\frac{30^\circ}{2^{12}} = \sqrt{\frac{1-C_{11}}{2}} = 0{\cdot}00012783$$

from which it appears, that of small arcs the approximate values of the sines are halved, when the arcs are bisected, or in other words, these values are proportional to the arcs; hence,

$$\frac{30^\circ}{(2)^{11}} : \frac{1^\circ}{60} :: 0\cdot00025566 : 0\cdot00029088 =$$ sine of one minute;

also,

cos of one minute $= \sqrt{1 - \sin^2 1'} = 0\cdot999999957$.

(117.) We may also determine the value of sin 1′ by means of the formula in (24).

$$\sin A = \tfrac{1}{2}\{\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}\}.$$

For let $2A = 30^\circ$, $\therefore \sin 2A = \frac{1}{2}$, and

$$\sin\frac{30^\circ}{2} = \tfrac{1}{2}(\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}}) = S_1 \text{ (suppose)}$$

$$\sin\frac{30^\circ}{2^2} = \tfrac{1}{2}(\sqrt{1+S_1} - \sqrt{1-S_1}) = S_2$$

$$\sin\frac{30^\circ}{2^3} = \tfrac{1}{2}(\sqrt{1+S_2} - \sqrt{1-S_2}) = S_3$$

proceeding in this manner, we shall have, by computation,

$$\sin\frac{30^\circ}{2^{11}} = \tfrac{1}{2}(\sqrt{1+S_{10}} - \sqrt{1-S_{10}})$$
$$= 0\cdot0002556634;$$

from which we obtain,

$$\sin 1' = 0{\cdot}0002908882$$

and thence, $\cos 1' = 0{\cdot}999999957$.

(118.) This method is more accurate than the former; for in the equation

$$\sin A = \sqrt{\frac{1 - \cos 2A}{2}},$$

let $\frac{1 - \cos 2A}{2} = v$, and suppose x to represent the error, or the value of the figures neglected; then, $\sin A = \sqrt{v + x}$ accurately $=$ (by developing) $v^{\frac{1}{2}} + \frac{x}{2\sqrt{v}} + \ldots\ldots$ From which it appears that the value of the sine is liable to the error $\frac{x}{2\sqrt{v}}$ nearly, which, when v is small, is very considerable. It was owing to this that we were unable to determine the value of $\sin \frac{30^{\circ}}{2^{11}}$ (in 116.) to the same accuracy as that of the cosine.

(119.) We are now enabled to deduce the values of the sine and cosine of 2′, 3′, 4′, &c.

From (26.), we have,

$$\sin(n+1)A = 2\cos A \cdot \sin nA - \sin(n-1)A;$$

suppose $A = 1'$, and $n = 1, 2, 3$, &c., we shall have, successively,

$$\begin{aligned} \sin 2' &= 2\cos 1' \cdot \sin 1' \\ \sin 3' &= 2\cos 1' \cdot \sin 2' - \sin 1' \\ \sin 4' &= 2\cos 1' \cdot \sin 3' - \sin 2' \\ \&c. &= \quad \&c. \end{aligned}$$

From (27.), we have,

$$\cos(n+1)A = 2\cos A \cdot \cos nA - \cos(n-1)A;$$

let $A = 1'$, $n = 1, 2, 3$, &c., successively,

$$\begin{aligned} \cos 2' &= 2\cos 1' \cdot \cos 1' - 1 \\ \cos 3' &= 2\cos 1' \cdot \cos 2' - \cos 1' \\ \cos 4' &= 2\cos 1' \cdot \cos 3' - \cos 2' \\ \&c. &= \quad \&c. \end{aligned}$$

Proceeding in this manner, the sines and cosines may be carried as far as 30°; after which the labour is, in a great measure, at an end.

(120.) By (21.),

$$\sin(A+B) = 2\sin A \cdot \cos B - \sin(A-B),$$

let $A=30^\circ$ or $\sin A=\frac{1}{2}$, and $B=1', 2', 3'$, &c., successively,

$$\begin{aligned} \sin 30^\circ\, 1' &= \cos 1' - \sin 29^\circ\, 59', \\ \sin 30^\circ\, 2' &= \cos 2' - \sin 29^\circ\, 58', \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Again,

$$\cos (A+B) = \cos (A-B) - 2 \sin A \,.\, \sin B,$$

making the same substitutions as before,

$$\begin{aligned} \cos 30^\circ\, 1' &= \cos 29^\circ\, 59' - \sin 1', \\ \cos 30^\circ\, 2' &= \cos 29^\circ\, 58' - \sin 2', \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Proceeding in this way to 45°, we shall have obtained the values of the sines and cosines of every minute in the quadrant.

(121.) There is another method of computing the sines.

By (21.),

$$\sin (A+B) = 2 \sin A \,.\, \cos B - \sin (A-B)$$

$$= 2 \sin A \left(1-2 \sin^2 \frac{B}{2}\right) - \sin (A-B)$$

$$= 2 \sin A - \sin (A-B) - 4 \sin^2 \frac{B}{2} \,.\, \sin A.$$

Suppose $B=1'$, and $A=1'$, $2'$, $3'$, &c., successively, we shall have,

$$\begin{aligned}
\sin 2' &= 2\sin 1' - 4\sin^2 30'' \,.\, \sin 1' \\
\sin 3' &= 2\sin 2' - \sin 1' - 4\sin^2 30'' \,.\, \sin 2' \\
\sin 4' &= 2\sin 3' - \sin 2' - 4\sin^2 30'' \,.\, \sin 3' \\
\&c. &= \qquad \&c.
\end{aligned}$$

Since $4 \sin^2 30'' = 0{\cdot}0000000846$ is a common multiplier, a table of its products by the nine digits may be previously made, which will greatly facilitate the calculation.

(122.) Proceeding by the process here explained, as far as 60°, we may make use of a particular formula for the calculation of those up to 90°.

We have in (40.),

$$\sin(60+A)=\sin(60-A)+\sin A,$$

let $A=1'$, $2'$, $3'$, &c., successively, then,

$$\begin{aligned}
\sin 60^\circ\, 1' &= \sin 59^\circ\, 59' + \sin 1 \\
\sin 60^\circ\, 2' &= \sin 59^\circ\, 58' + \sin 2 \\
\&c. &= \qquad \&c.
\end{aligned}$$

The sines being calculated, the cosines will be known from the formula $\cos A=\sin\left(\frac{\pi}{2}-A\right)$.

(123.) In performing so many long and laborious operations, errors are very liable to be made, which will render erroneous all the succeeding tabulated results. It is therefore necessary at certain intervals to calculate their values independently of the method used, which will serve as verifications of all the preceding results. The most convenient for this purpose is the method following.

(124.) We have found in (104.) for sin x, the series

$$x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}$$
$$+\&c.$$

and for cos x, the series

$$1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\&c.$$

let $x=\frac{m}{n}\cdot\frac{\pi}{2}$; then

$$\sin\frac{m}{n}\cdot\frac{\pi}{2}=\frac{m}{n}\cdot\frac{\pi}{2}-\frac{m^3}{n^3}\cdot\frac{1}{1.2.3}\left(\frac{\pi}{2}\right)^3$$

$$+\frac{m^5}{n^5}\cdot\frac{1}{1.2.3.4.5}\left(\frac{\pi}{2}\right)^5-\&c.$$

$$\cos\frac{m}{n}\cdot\frac{\pi}{2}=1-\frac{m^2}{n^2}\,\frac{1}{1.2}\left(\frac{\pi}{2}\right)^2$$

$$+\frac{m^4}{n^4}\cdot\frac{1}{1\,.\,2\,.\,3\,.\,4}\left(\frac{\pi}{2}\right)^4-\ \&c.$$

Now, (since $\pi = 3\cdot1415926535897932$) if the values of

$$\frac{\pi}{2},\ \frac{1}{1\,.\,2}\left(\frac{\pi}{2}\right)^2,\ \frac{1}{1\,.\,2\,.\,3}\left(\frac{\pi}{2}\right)^3,\ \frac{1}{1\,.\,2\,.\,3\,.\,4}\left(\frac{\pi}{2}\right)^4\ \&c.$$

be expressed in decimals, we shall have,

$$\sin\frac{m}{n}\cdot\frac{\pi}{2}=$$

$$\begin{array}{lllll}
+\dfrac{m}{n} & \times\ 1\cdot57079 & 63267 & 948966 \\
+\dfrac{m^5}{n^5} & \times\ 0\cdot07969 & 26262 & 461670 \\
+\dfrac{m^9}{n^9} & \times\ 0\cdot00016 & 04411 & 847874 \\
+\dfrac{m^{13}}{n^{13}} & \times\ 0\cdot00000 & 00569 & 217292 \\
+\dfrac{m^{17}}{n^{17}} & \times\ 0\cdot00000 & 00000 & 060669 \\
+\dfrac{m^{21}}{n^{21}} & \times\ 0\cdot00000 & 00000 & 000000 \\
 & & & \\
-\dfrac{m^3}{n^3} & \times\ 0\cdot64596 & 40975 & 062463 \\
-\dfrac{m^7}{n^7} & \times\ 0\cdot00468 & 17541 & 353187
\end{array}$$

$$-\frac{m^{11}}{n^{11}} \times 0\cdot00000 \quad 35988 \quad 432352$$

$$-\frac{m^{15}}{n^{15}} \times 0\cdot00000 \quad 00006 \quad 688035$$

$$-\frac{m^{19}}{n^{19}} \times 0\cdot00000 \quad 00000 \quad 000438$$

$$-\frac{m^{23}}{n^{23}} \times 0\cdot00000 \quad 00000 \quad 000000$$

$$\cos\frac{m}{n}\cdot\frac{\pi}{2} =$$

$$1\cdot00000 \quad 00000 \quad 000000$$

$$+\frac{m^{4}}{n^{4}} \times 0\cdot25366 \quad 95079 \quad 010480$$

$$+\frac{m^{8}}{n^{8}} \times 0\cdot00091 \quad 92602 \quad 748394$$

$$+\frac{m^{12}}{n^{12}} \times 0\cdot00000 \quad 04710 \quad 874779$$

$$+\frac{m^{16}}{n^{16}} \times 0\cdot00000 \quad 00000 \quad 656596$$

$$+\frac{m^{20}}{n^{20}} \times 0\cdot00000 \quad 00000 \quad 000034$$

$$-\frac{m^{2}}{n^{2}} \times 1\cdot23370 \quad 05501 \quad 361698$$

$$-\frac{m^{6}}{n^{6}} \times 0\cdot02086 \quad 34807 \quad 633530$$

$$-\frac{m^{10}}{n^{10}} \times 0\cdot00002\quad 52020\quad 423731$$

$$-\frac{m^{14}}{n^{14}} \times 0\cdot00000\quad 00063\quad 866031$$

$$-\frac{m^{18}}{n^{18}} \times 0\cdot00000\quad 00000\quad 005294$$

$$-\frac{m^{22}}{n^{22}} \times 0\cdot00000\quad 00000\quad 000000$$

If we assume $m=1$ and $n=5400$, we shall obtain the sine and cosine of one minute true to fifteen decimals, by using only two terms of each series. And by giving to m and n the proper values, we may calculate the sines and cosines of arcs at convenient intervals. If these are found to agree with the corresponding results obtained by the continuous process, it is scarcely possible to have a better verification.

(125.) The tangents are immediately drawn from the relation, $\tan A = \frac{\sin A}{\cos A}$. These being calculated up to 45°, the formula

$$\tan(45^\circ + A) = 2 \tan 2A + \tan(45^\circ - A)$$

enables us to obtain those up to 90° with great facility: thus, making A successively $=1^\circ, 2^\circ, 3^\circ$, &c.,

$$\begin{aligned}\tan 46^\circ &= 2\tan 2^\circ + \tan 44^\circ\\ \tan 47^\circ &= 2\tan 4^\circ + \tan 43^\circ\\ \tan 48^\circ &= 2\tan 6^\circ + \tan 42^\circ\\ \&c. &= \quad \&c.\end{aligned}$$

Hence the tangents and cotangents are known.

(126.) The secants are calculated from the formula

$$\tan A + \cot A = 2 \operatorname{cosec} 2A\,^*.$$

This gives the secants or cosecants only for every second division: the intermediate ones may be obtained from the relation

$$\sec A = \frac{1}{\cos A}, \text{ or } \operatorname{cosec} A = \frac{1}{\sin A}.$$

(127.) Thus the table of natural sines, tangents, and secants, is completed. The logarithmic table of sines, &c., may be formed by taking from the table

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[*] The truth of this is readily shewn, thus:

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{1}{\sin A \cdot \cos A} = \frac{2}{2\sin A \cdot \cos A}$$

$$= \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A.$$

of logarithms, the logarithms of these numbers, and Cagnoli and others have thought this, upon the whole, the best method. They may, however, be calculated independently of the natural sines and of logarithmic tables, and therefore free from all errors of previous computations; but as this depends upon reasoning totally unsuitable to an elementary work like the present, it would be foreign to our purpose to treat of it here (*d*).

ELEMENTS OF ASTRONOMY.

SECTION X.

DOCTRINE OF THE SPHERE.

(128.) If, on a clear night *, in a place where the view of the horizon is uninterrupted, the heavens be attentively observed, it will be seen that changes are continually taking place. Some stars rise above the horizon towards the east, some disappear towards the west, and others never reach the horizon at all; whilst their relative position is constantly the same. Towards morning they become fainter, and at last disappear on the approach of the sun; their light being overpowered by the more vivid rays of this luminary. Other stars, though invisible to the naked eye, are perpetually rising and setting during the day. Thus all the heavenly bodies appear to describe circles about an imaginary axis, called from this circumstance the *axis of the world*, the extremities of this axis being denominated the *poles of the world.* This appearance is caused by the rotation of the earth on an axis from west to east; hence every point of the heavens will seem to describe from east to west great or small circles, the centres of which are situated in the axis of the world. The pole elevated above the horizon in our climate, is called the *north pole,* and that depressed below the horizon, the *south pole.* The axis of the world coinciding with the axis of the earth, passes through its north and south poles. The great circle perpendicular to the axis of the world, is called the *celestial equator,* the plane of which coincides with that of the terrestrial equator. The small circles parallel to the equator are called *parallel circles.*

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* Vide *System of the World*, by Laplace.

(129.) The *zenith* is the point in which a vertical to the surface of the earth at the place of the observer meets the celestial sphere: the opposite point is called the *nadir.*

(130.) The *meridian* is the great circle which passes through the zenith and poles; it divides into two equal parts the arcs described by the heavenly bodies above the horizon; so that when they are upon this circle they are at their greatest or least altitude.

(131.) The horizon is the great circle perpendicular to the vertical, or parallel to the surface of stagnant water at the place of the observer, which surface is called the *sensible* or *artificial* horizon.

(132.) The great circle which passes through the zenith perpendicular to the meridian, is called the *prime vertical:* it cuts the horizon in the east and west points.

(133.) Great circles passing through the poles are called *hour circles;* and that hour circle which is perpendicular to the meridian is called the *six o'clock hour circle.*

(134.) The *ecliptic* is the great circle which the sun describes in its apparent annual motion round the earth. It intersects the equator in two points, called the *vernal* and *autumnal equinoxes;* because when the sun is in these points, the days and nights are equal in all parts of the earth.

(135.) The spherical angle made by the intersection of these two circles, or the inclination of their planes, is called the *obliquity of the ecliptic.*

(136.) A great circle passing through the sun and the poles, will intersect the equator in a point which represents the place of the sun referred to the equator. The arc of the equator intercepted between this point and the vernal equinox, is called the ***sun's right ascension.*** The arc of the ecliptic intercepted between the sun and the same equinox, is denominated the ***sun's longitude.*** And the arc of the great circle intercepted between the sun and its place referred to the equator, is called the ***sun's declination.***

(137.) The longitude and right ascension are measured in the direction of the proper motion of the sun. When the sun is in that part of the ecliptic which is towards the north pole, the declination is called ***north,*** and when in that part which is towards the south pole, the declination is ***south.***

(138.) The variety of seasons is caused by the inclination of the equator to the ecliptic. When the sun, in his annual progress, arrives at the equator, he describes in his diurnal motion, this great circle, very nearly, which being divided into two equal parts by all the horizons, the day is equal to the night in every part of the earth. As the sun, after leaving the vernal equinox advances in his orbit towards the north, his meridian altitudes above our horizon increase, the visible arc of the parallels which he describes every day, continually increases, and this augments the length of the day, till the sun has attained his greatest declination. At this time, the days are the longest in the year; and because the motion of the sun is then perpendicular to the meridian, the variation in his meridian altitudes is insensible for several days together, and the sun appears stationary; hence this point has been termed the *summer solstice,* and the parallel described by the sun on that day, the *summer tropic.* The sun then descends towards the equator, which he crosses at the autumnal equinox, and thence he arrives at the greatest southern declination, or *winter solstice:* his meridian altitude is then the least, the parallel described by the sun is the *winter tropic,* and the corresponding day, the shortest in the year. Having attained this term, the sun again ascends and returns to the equinox of spring to recommence the same route. The equinoxes take place about the 21st of March and the 23rd of September. The solstices about the 21st of June and the 22nd of December. These epochs are, respectively, the commencement of *Spring* and *Autumn, Summer* and *Winter.*

(139.) Let C (fig. 27.) represent the position of the earth, $P\ S$ the axis of the world, and $P\ E\ S\ Q$ the meridian of a spectator situated upon the earth at b. Since the prime vertical, equator, horizon and the six o'clock hour circle are all perpendicular to the meridian, if we project these circles upon the plane of the meridian, they will be represented by the right lines *Z N*, *E Q*, *H R* and *P S*, respectively.. Now, if we consider the magnitude of the celestial sphere, embracing the visible fixed stars, we may reasonably be allowed to represent the earth by a point at *C*, and, in further inquiries on this subject, may altogether leave its magnitude out of consideration: the relative position of the spectator being known from those of zenith and pole of the world.

(140.) When the sun is in the equator, he will, by the diurnal motion, describe this circle (neglecting his change in declination). Hence, if figure 27, represent the eastern hemisphere, at midnight the sun will be at *Q*; at six o'clock he will rise above the horizon in the east at *C*; and at noon will have attained the meridian at *E*. He then descends in the western hemisphere, and sets in the west point of the horizon. If the sun be on the north side of the equator *, or the declination be north, he will describe a small circle, which being projected upon the plane of the meridian, may be represented by the arc *m n*. After leaving the position, *m*, at midnight, he will rise at *p*, at six o'clock will be at

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* Suppose in the ecliptic at *o*.

q, and then crossing the prime vertical at *r*, will ascend to the meridian at *n*. When the declination is south, the sun will traverse the prime vertical and six o'clock hour circle before he rises. This position of the sphere is called an *oblique sphere.*

(141.) If the situation of the observer be north of the equator, and such that the co-altitude of the pole be less than the declination, the sun will not descend below the horizon, nor ascend above it, according as the declination is north or south. And under the pole, the horizon coinciding with the equator, the sun will be perpetually visible and invisible, alternately, for half a year; such a position of the sphere is denominated a *parallel sphere.*

(142.) To an observer at the equator, the poles being in the horizon, all the stars rise and set perpendicular to this circle; the day is constantly equal to the night, and at the equinoxes, the sun passes through the zenith at mid-day. The position is called a *right sphere.*

(143.) Having explained what is called the *doctrine of the sphere,* we shall proceed to the solution of astronomical problems.

SECTION XI.

ASTRONOMICAL PROBLEMS.

(144.) Problem I. To find the obliquity of the ecliptic.

Let $E\,Q$ (fig. 27.) represent the equator, and $W\,s$ the ecliptic[*]; then at the solstices the sun's declination is the measure of the obliquity: suppose the sun at these times to describe the parallels $s\,S'$, $w\,W$; then the meridian altitudes are $H\,S'$, $H\,W$; hence $E\,S' = \frac{H\,S' - H\,W}{2}$; that is, *the inclination of the equator to the ecliptic is equal to half the difference of the meridian altitudes of the sun at the summer and winter solstices.*

(145.) From the earliest observations, it appears that this angle has been diminishing. The following table, given by Laplace, clearly points out the successive diminution in an interval of 2900 years.

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[*] Projected upon the plane of the meridian when the equinoxes are in the horizon.

Tcheou King	B.C.	1100	23°	54′	2″
Pythias		350	23	49	20
Ebn Junis	A.D.	1000	23	34	26
Cocheou King		1280	23	32	2
Ulug Beigh		1437	23	31	48
In the year		1801	23	27	57

The mean obliquity, Jan. 1, 1832, was 23° 27′ 41″·73, and it is diminishing at the rate of about 0″·48[*] annually. The profound investigations of Laplace in the *Celestial Mechanics,* have shewn that this diminution has a limit, and that the whole amount of variation is about 2° 40′.

(146.) This phenomenon is connected in its cause with another, which is known by the term *precession of the equinoxes.* It is found that the sun returns to the equinoxes in less time than it returns to the same fixed star, or to the same point in the heavens; the equinoxes have, therefore, a retrograde motion with respect to that of the sun. The effect of this is, that the longitude[†] of the stars, which is estimated from the vernal equinox, constantly increases; and by comparing the longitude of the same stars, determined at distant intervals, the mean annual motion of these points, or the precession of the equinoxes, is found to be about $50'' \cdot 1$.

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[*] Con. des Temps. 1833.

[†] Vide (163.)

(147.) Problem II. Given the sun's longitude, and the obliquity of the ecliptic to find the right ascension and declination.

Suppose E (fig. 28.) to represent the position of the earth, A the vernal equinox, $A\,C\,L$ the equator, $A\,B\,L$ the ecliptic, P the north pole, and B the place of the sun. Draw the great circle $P\,B\,C$, intersecting the equator in C: then $A\,B$ is the sun's longitude, $A\,C$ the right ascension, $B\,C$ the declination, and the angle A the obliquity of the ecliptic (135.) (136.)

In the triangle $A\,B\,C$, right angled at C, we have,

$$\tan b = \frac{\cos A}{\cot c} \ldots\ldots(1.)$$

$$\sin a = \sin A \, . \, \sin c \ldots\ldots(2.)$$

These equations, owing to the unrestricted definition of a spherical triangle, are general, whatever be the magnitude of c, the sun's longitude.

1°. When $c = \frac{\pi}{2}$, $\cot c = 0 \; \therefore \tan b = \infty$, or $b = \frac{\pi}{2}$.

Also,

$$\sin a = \sin A, \text{ or } a = A.$$

This corresponds to the summer solstice.

2°. When $c = \pi$, $\cot c = \infty$, and $\sin c = 0$,

$$\therefore \tan b = 0 \text{ or } b = \pi.$$

And,

$$\sin a = 0, \text{ or } a = 0,$$

corresponding to the autumnal equinox.

3°. If $c = \dfrac{3\pi}{2}$, $\cot c = 0$, and $\sin c = -1$.

$$\therefore \tan b = \infty, \text{ or } b = \frac{3\pi}{2} \text{ and } \sin a = -\sin A,$$

shewing that the declination is south, and corresponding to the winter solstice.

4°. When $c = 2\pi$, $b = 2\pi$ and $a = 0$,

as they ought to be at the vernal equinox. The longitude at this instant commences again.

It is more convenient, however, in solving this problem, to apply the equations (1.), (2.) to the triangles, $A B C$, $L B C$, $L B' C'$ and $A B' C'$, according as the longitude is less than $\dfrac{\pi}{2}$, π, $\dfrac{3\pi}{2}$ and 2π respectively.

By this problem, the columns of the sun's right ascension* (A.R) and declination (Dec.) in the

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* Right ascension being generally given in *time*, we shall shew in this place, how degrees, &c., are converted into time, and the contrary.

As 15° are equivalent to one hour
1° is .... 4^m minutes of time
1′ .... 4^s seconds
1″ .... 4^{th} thirds.

If then the seconds of space be multiplied by 4 the result will be thirds of time. Similarly the minutes produce *seconds*, and the degrees, *minutes* of time.

Let it be required to convert 23° 26′ 11″ into time: the operation will stand thus :—

23° 26′ 11″
4
$1^h\ 33^m\ 44^s\ 44^{th}$.

To convert time into degrees, &c., we have only to multiply by 15.

Convert $1^h\ 33^m\ 44^s\ 44^{th}$ into degrees.

$1^h\ 33^m\ 44^s\ 44^{th}$
10
15 37 27 20
one half 7 48 43 40
23° 26′ 11″ corresponding time.

Nautical Almanac are calculated. In order to make the results agree with observation, the *apparent*, or *true* obliquity must be obtained. The true obliquity is that which results from taking into account the precession of the equinoxes combined with other perturbating forces: it is given in the Almanac for every three months, and for any intermediate day, it is obtained by interpolating. In the Almanacs for 1834 and succeeding years, it will be given for every ten days. The true obliquity of the ecliptic for Jan. 1, 1832, was 23° 27′ 34″ · 6.

Example I. At apparent noon at Greenwich, Aug. 1, 1832,

☉'s long.$=4^{s}$ 9° 9′ · 8″, true obliq.$=$23° 27′ 36″.

Required the right ascension, and declination.

Answer, A.R$=8^{h}\ 46^{m}\ 21^{s}\cdot 9$; Dec.$=$17° 58′ 56″ N.

Example II. At noon, Jan. 18, 1833,

☉'s long.$=9^{s}$ 28° 14′ 51″, obliq.$=$23° 27′ 36″ · 8.

Required the right ascension and declination.

Answer, A.R$=20^{h}\ 1^{m}\ 25^{s}\cdot 6$; Dec.$=$20° 31′ 49″ S.

Example III. At noon, June 10, 1833.

$\odot$'s long.$=2^{s}\ 19^{\circ}\ 17'\ 11''$, obliq.$=23^{\circ}\ 27'\ 38''$.

Required the right ascension and declination.

Answer, A.R$=5^{h}\ 13^{m}\ 23^{s}$; Dec.$=23^{\circ}\ 1'\ 40''$ N.

Example IV. At noon, Dec. 31, 1833.

$\odot$'s long.$=9^{s}\ 9^{\circ}\ 39'\ 26''$; Dec.$=23^{\circ}\ 6'\ 33''$ S.

Required the true obliquity and right ascension.

Answer, A.R$=18^{h}\ 42^{m}\ 21^{s}$; obliq.$=23^{\circ}\ 27'\ 40''$.

(148.) Problem III. To find the altitude of the pole above the horizon.

Let the greatest and least altitudes of a star, which does not set, be observed; if these be Rx, Ry (fig. 27.) we shall have,

$$Rx+Ry=2\,PR \;\therefore\; PR=\frac{Rx+Ry}{2}.$$

(149.) Definition: The *latitude* of a place upon the earth is its angular distance from the equator; or it may otherwise be defined to be, the arc of the meridian intercepted between the equator and the zenith of the observer.

(150.) Hence the altitude of the pole is equal to the latitude. For $ZP+PR$ (fig. 27.) $=90°$ $=ZP+EZ$.

$$\therefore PR=EZ=\text{latitude}.$$

(151.) Problem IV. Given the true meridian altitude of the sun's centre*, and the declination, to find the latitude.

Let n (fig. 27.) be the place of the sun when on the meridian (the declination being north), then Hn is the meridian altitude, and $90°-Hn=Zn$ the zenith distance: if to this we add En the declination, we have ZE the latitude.

When the declination is south, the method is similar and equally easy.

(152.) Problem V. Given the sun's altitude, the declination, and the hour of the day, to determine the latitude.

In the spherical triangle $ZP\odot$ (fig. 27.) where Z is the zenith, P the pole, and $\odot$ the place of the

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[*] This is also to be understood in all the succeeding problems: the altitudes of the heavenly bodies being given, corrected of all the irregularities with which they are entangled in practice, before they are submitted to calculation.

sun, we have given $Z\odot$ the complement of the altitude, or *zenith distance,* $P\odot$ the complement of the declination, when north (or 90°+declination when south), *the polar distance,* and the angle $ZP\odot$ the horary angle from noon, to find ZP the complement of the latitude.

(153.) Problem VI. Given two altitudes of the sun, the declination at each altitude, and the interval of time between the observations, to find the latitude.

Let Z (fig. 29.) represent the zenith, P the pole, and A, B, the places of the sun at the times of observation.

In the triangle ABP, the two sides PB, PA are known, and the angle APB (corresponding to the interval of time), to find the angle PAB, and the side AB. In the triangle ABZ the three sides are given to find the angle BAZ: then the angle ZAP will be known. Lastly, in the triangle AZP, the two sides AZ and AP and the included angle are known, to find the third side ZP the co-latitude.

(154.) Problem VII. Given the altitudes of two fixed stars observed at the same time, their polar distances and their difference of right ascension [*], to find the latitude.

The solution of this problem depends upon principles precisely similar to those of the last; the difference of right ascension corresponding to the interval of time.

EXAMPLES.

1. At sea on the 5th of Dec. 1823, the declination being 22° 18′ 14″ south, the sun's meridian altitude was found to be 25° 38′ 30″; what was the latitude of the ship?

Answer, 42° 3′ 16″ north.

2. Suppose two altitudes of the sun to be 16° 5′ 47″ and 42° 14′ 9″, the corresponding declinations 8° 18′ and 8° 15′ north, and the interval of time three hours; to determine the latitude.

Answer, 48° 50′.

3. Let the corrected zenith distance of Capella be 29° 14′ 24″$=ZA$ (fig. 29.), and that of Sirius be 72° 5′ 48″$=ZB$. The polar distance of Capella $PA=$44° 11′ 39″, that of Sirius $PB=$106° 28′ 40″,

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[*] Vid. (163.)

and the difference of their right ascensions $1^h\ 33^m\ 45^s$, or $APB = 23°\ 26'\ 11''$. Required the latitude.

Answer, 53° 19′ 22″.

(155.) Problem VIII. Given the latitude of the place, the sun's altitude and declination, to find the hour.

In the spherical triangle $ZP\odot$ (fig. 27.) the three sides are given to find the angle $ZP\odot$, which corresponds to the time from apparent noon.

Example. Suppose the latitude to be 34° 55′, the sun's altitude 36° 59′ 39″, and the declination 22° 22 57″ N., to determine the time.

Answer, $3^h\ 58^m\ 22^s$ from apparent noon *.

(156.) Problem IX. Given the latitude of the place (l), and the sun's declination (d) to find the time when the sun rises.

Let the declination be north, and A (fig. 30.) the position of the sun on the horizon at the time required. Draw the hour circle, PAB, intersecting the equator in B: then in the spherical triangle ABC, right angled at B, we have given AB = declination, and the angle ACB = co-latitude, to find

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* Vid. (167.)

the side BC, the measure of the angle CPB, which corresponds to the time before six o'clock.

By Napier's rules, we have

$$\sin BC = \cot ACB \,.\, \tan AB$$
$$= \tan l \,.\, \tan d.$$

If the declination be south, let A' be the sun's position when rising; then in triangle $A'B'C$ we shall likewise have,

$$\sin B'C = \tan l \,.\, \tan d.$$

Hence the interval of time between sun-rising and six o'clock becomes known.

The same formula may be found from the solution of the quadrantal triangle APZ, or $A'PZ$.

By (88.) we have,

$$\cos APZ = \frac{\cos 90^\circ - \cos PA \,.\, \cos PZ}{\sin PA \,.\, \sin PZ}$$
$$= -\cot PA \,.\, \cot PZ;$$

but, $\cos APZ = \sin\left(\frac{\pi}{2} - APZ\right) = \sin - APC = -\sin APC$ (9.)

also,

$$\cot PA = \tan d, \text{ and } \cot PZ = \tan l.$$

$$\therefore \sin APC = \tan d \,.\, \tan l.$$

If h denotes the horary angle from noon when the sun is rising or setting, we have,

$$\cos h = \tan l \,.\, \tan d;$$

when the latitude and declination are of the same denomination, $\cos h$ is negative, or h is greater than $90°$ $\therefore$ the day is greater than the night; when l or d, or both, vanish, $h = 90°$, therefore the day is always equal to the night; and when $\tan l = \cot d$, or $l = 90° - d$, $h = 0$, hence the sun does not set.

Example I. At what time on the longest day in 1832 did the sun rise at London, in latitude 51° 32′?

tan 51° 32′, log..........	10 · 0999135
tan 23° 27′ 36″..........	9 · 6374721
sin 33° 6′ 35″............	9 · 7373856

This converted into time shews that the sun rose $2^{h}\ 12^{m}\ 26^{s}$ before six o'clock.

Example II. When did the sun rise at Edinburgh, in latitude 55° 56′ 42″ N., on the 1st of December, 1832, the declination being 21° 52′ 24″ south?

Answer, $2^{h}\ 25^{m}\ 45^{s}$ after six o'clock.

Example III. What diminution has the length of the longest day undergone in the latitude of London since the year A.D. 1000?

Answer, $1^m\ 38^s$.

(157.) Problem X. Given the latitude (*N*), and the sun's declination (*N*), to find the altitude of the sun at six o'clock.

In the right angled triangle $Z\,P\,\odot$[*], of which the angular points are the zenith, pole, and sun, we have given $Z\,P$, the co-latitude, $P\,\odot$ the co-declination, to find $Z\,\odot$, the co-altitude.

Hence,

$$\cos Z\,\odot = \cos Z\,P \,.\, \cos P\,\odot,$$

or,

$$\sin \text{altitude} = \sin l \,.\, \sin d.$$

Example. At Vienna, in latitude 48° 12′ 40″ N. what was the sun's altitude at six o'clock on the 4th of April, 1832; the declination being 5° 48′ 27″ N.

Answer, 4° 19′ 36″.

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[*] To this problem and the two following ones, figures, if necessary, may readily be supplied.

(158.) Problem XI. Given the same as in the last, to find the altitude of the sun when due east, and the hour.

When the sun is in the east, he is upon the prime vertical. In the triangle $Z P \odot$, right angled at Z, we have $Z P$, $P \odot$ given to find $Z \odot$ the co-altitude, and $Z P \odot$ the horary angle. Hence,

$$\cos P \odot = \cos Z P \,.\, \cos Z \odot.$$

$$\therefore \sin \text{altitude} = \frac{\sin d}{\sin l}.$$

Also,

$$\cos Z P \odot = \tan Z P \,.\, \cot P \odot,$$

or,

$$\cos h = \cot l \,.\, \tan d.$$

If the latitude and declination be of different denominations, *altitude* must be changed to *depression*; and cos h will be negative, or h must be taken greater than 90°.

Example I. At Greenwich, in latitude 51° 28′ 40″ N., on the 4th of April, 1832, the declination being 5° 48′ 27″ N.; required the altitude of the sun when due east, and the hour.

Altitude = 7° 25′ 52″.

Time = 18^{m} 35^{s} after six o'clock.

Example II. At what hour (Greenwich time) on the 29th of May, 1833, the declination being 21° 37′ 36″ N., will the sun be due west at the Cape of Good Hope, in latitude 33° 55′ 42″ S., longitude 10° 23′ E.; and what will be the sun's depression at that time?

Time $=7^h\ 43^m\ 57^s$ P.M. Depression $=41°\ 19'\ 25''$.

The above declination corresponds to the time of apparent noon at the Cape.

(159.) Definitions: The *azimuth* of a heavenly body is the angle at the zenith formed by the meridian and the vertical circle passing through the body.

The *amplitude* is the angle at the zenith formed by the prime vertical, and the same vertical circle.

(160.) Problem XII. Given the latitude, the sun's declination and altitude, to find the sun's azimuth.

In the spherical $Z\,P\,\odot$, the three sides are given to find the angle Z, the azimuth *from the north.*

Example. Suppose the altitude of the sun at Greenwich should be observed to be 18° 44′ on Nov.

16, 1833, what will be its azimuth; the declination for that day being 18° 46′ 23″ s.?

$$\begin{array}{rr}
 & 90^\circ \\
 & 51\ 28\ 40 \\
\hline
ZP= & 38\ 31\ 20
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 90^\circ \\
 & 18\ 44 \\
\hline
Z\odot= & 71\ 16
\end{array}$$

$$\begin{array}{rr}
 & 90^\circ \\
 & 18\ 46\ 23 \\
\hline
P\odot= & 108\ 46\ 23
\end{array}$$

Calling the above three sides, *a*, *b*, *c*, respectively, we shall obtain the azimuth by computing the formula,

$$\sin\frac{C}{2}=\sqrt{\frac{\sin\overline{S-a}\,.\,\sin\overline{S-b}}{\sin a\,.\,\sin b}}.$$

$$\begin{array}{rl}
a= & 38^\circ\ 31'\ 20'' \\
b= & 71\ \ 16\ \ 00 \\
c= & 108\ \ 46\ \ 23 \\
\hline
2) & 218\ \ 33\ \ 43 \\
\hline
S= & 109\ \ 16\ \ 51\cdot5
\end{array}$$

$$\begin{array}{rlll}
\sin\ldots & 38\ 31\ 20 & \text{co ar}\ \ldots\ldots & 0\cdot2056388 \\
\hline
S-a= & 70\ 45\ 31\cdot5 & \log\sin\ldots & 9\cdot9750361 \\
\hline
\end{array}$$

	$109^\circ\ 16'\ 51''{\cdot}5$	
sin...	$71\ \ 16$	co ar..... $0{\cdot}0236392$
$S-b=$	$38\ \ 0\ \ 51\cdot5$	log sin... $9{\cdot}7894807$
		$2)19{\cdot}9937948$
$\frac{C}{2}=$	$83^\circ\ 9'\ 34''$	log sin... $9{\cdot}9968974$
	2	
$C=$	$166\ \ 19\ \ 8=PZ\odot$	

which is the azimuth from the north.

(161.) Def. *Twilight* is that faint light which appears in the horizon before the sun rises and after he sets. It is generally supposed to begin in the morning, and end in the evening, when the sun is 18 degrees below the horizon.

(162.) Problem XIII. Given the latitude and the sun's declination, to determine the time when the twilight begins and ends.

In the spherical triangle $ZP\odot$ (fig. 31.) we have given $ZP=$ co-latitude, $P\odot=$ co-declination when north (or $=90^\circ+$ declination when south) and $Z\odot=108^\circ$, to find the angle $ZP\odot$, the horary angle.

Example. At London, in latitude 51° 32′, when the sun's declination is 11° 46′ s., at what time did the twilight begin?

Answer 3^m 12^s after five o'clock.

(163.) Problem XIV. To find the time when the apparent diurnal motion of a star is perpendicular to the horizon.

Let $x\,y$ (fig. 31.) be the parallel described by the star, and suppose the vertical circle $Z\,h$ to be drawn touching it at S: now when the star arrives at this point, its motion is perpendicular to the horizon, being at that instant in the direction of the vertical $Z\,h$. Join $P\,S$, then $Z\,S\,P$ is a right angle, and hence we have,

$$\cos Z\,P\,S = \cot Z\,P \,.\, \tan P\,S.$$

$$= \tan l \,.\, \tan \text{polar dist. of star.}$$

Now the time when the star is upon the meridian, being supposed to be known, the angle $Z\,P\,S$ converted into time shews the interval to that time.

(164.) Definitions. Let S (fig. 32.) represent the position of a star, $A\,Q$ the equator, and $A\,L$ the ecliptic; draw the great circles $S\,B$, $S\,C$, perpendicular to $A\,Q$, $A\,L$, intersecting them respectively in B, C; then $A\,B$ is called the *right ascension* of the star, *S B* its *declination, A C* the star's *longitude,* and *S C* its *latitude.*

(165.) Problem XV. Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic, to find the latitude and longitude.

Referring to figure 32, draw the great circle *A S*; then in the right angled spherical triangle *A S B*, we have,

$$\tan SAB = \frac{\tan SB}{\sin AB} \ldots\ldots (1.)$$

hence the angle *S A B* becomes known, consequently also the angle *S A C*.

Again,

$$\cos AS = \cos AB \,.\, \cos SB \ldots\ldots (2.)$$

$$\therefore AS \text{ is given.}$$

Hence, in the triangle *S A C* right angled at *C*, we have,

$$\tan AC = \frac{\cos SAC}{\cot AS} \ldots\ldots (3.)$$

$$\sin SC = \sin AS \,.\, \sin SAC \ldots\ldots (4.)$$

therefore *A C* the longitude, and *S C* the latitude of the body become known.

Example. The right ascension of the fixed star Aldebaran, at midnight (Greenwich time) on Feb. 10th, 1834, $=66^\circ\ 35'\ 55''\cdot 1$, its declination $=16^\circ\ 10'\ 6''\cdot 2$ N., and the obliquity of the ecliptic $=23^\circ\ 27'\ 40''$. Required the latitude and longitude of the star.

Here, $AB=66^\circ 35'\ 55''\cdot 1$, $SB=16^\circ\ 10'\ 6''\cdot 2$, and $BAC=23^\circ\ 27'\ 40''$.

By equation (1.) we have,

$$\tan SAB=\frac{\tan SB}{\sin AB}$$

$$\begin{array}{lr} 10+\log\tan 16^\circ\ 10'\ \ 6''\cdot 2 \ldots\ldots & 19\cdot 4622912 \\ \log\sin\ 66^\circ\ 35'\ 55''\cdot 1 \ldots\ldots & 9\cdot 9627221 \\ \hline \log\tan SAB\ \ldots\ldots\ldots & 9\cdot 4995691 \end{array}$$

$$\therefore SAB=17^\circ\ 31'\ 55'', \text{ and}$$

$\therefore$ the star S is situated between the ecliptic and the equator, or in south latitude; hence the angle SAC $=5^\circ\ 55'\ 45''$.

Also, equ. (2.),

$$\cos AS=\cos AB\cdot\cos SB.$$

$$\begin{array}{lr} \log\cos 66^\circ\ 35'\ 55''\cdot 1 \ldots & 9\cdot 5989761 \\ \log\cos 16^\circ\ 10'\ \ 6''\cdot 2 \ldots & 9\cdot 9824736 \\ \hline \log\cos AS \ldots\ldots\ldots\ldots\ldots & 9\cdot 5814497 \end{array}$$

$$\therefore AS=67^\circ\ 34'\ 33''.$$

Again, by equ. (3.), (4.)

$$\tan AC = \tan AS \,.\, \cos SAC$$

$$\sin SC = \sin AS \,.\, \sin SAC.$$

log tan 67° 34′ 33″ ..... 10·3844036
log cos 5° 55′ 45″ .... 9·9976705

log tan AC .... 10·3820741

∴ $AC = 67° 28′ 2″$ the *'s longitude.

log sin 67° 34′ 33″ .... 9·9658530
log sin 5° 55′ 45″ .... 9·0140956

log sin SC .... 8·9799486

Hence, $SC = 5° 28′ 46″$ = the *'s latitude south.

The right ascensions and declinations of the stars are ascertained by observation, and hence the tables of the latitudes and longitudes have been computed.

(166.) As the obliquity of the ecliptic and the position of the vernal equinox are subject to continual change, the right ascensions, declinations, and longitudes of all the fixed stars are perpetually varying, principally from these causes. The longitude increases by a quantity nearly equal to the annual precession of the equinoxes; but the variations of the right ascensions, and declinations, owing to the combined influence of several elements, are not altogether uniform; some stars which had once north declination have now south declination, and *vice versâ*. The latitudes also, have a small annual variation.

(167.) The time which has been used in the preceding problems is *apparent time*, and not that which is shewn by a clock or chronometer. It may, perhaps, be proper in this place, to explain the difference. A *solar day* is the interval of time between two successive noons; it is greater than a revolution of the heavens, which constitutes a *sidereal day*; for if we imagine the sun and a star to pass the meridian at the same instant, the next day the sun will pass *later*, in consequence of his proper motion, from west to east. The solar days are not equal, owing to the obliquity of the ecliptic, and the unequal motion of the sun in the ecliptic. Hence we obtain the idea of a *mean solar day*:—the year being supposed to be divided into as many mean solar days as there are real days. The time measured by a clock so adjusted, is denominated *mean* or *true time*. The time shown by the sun is called *apparent time*; this is that which is used in the Nautical Almanac, and the difference between these is called the *equation of time*. The equation of time is given in the Almanac for every day in the year.

(168.) We have also supposed the heavenly bodies to rise and set when they are in the rational horizon; but they are really visible when they are 33′ below it; this effect, which is called *refraction*, diminishes from the horizon to the zenith, where it vanishes: it is the principal correction to be applied to the apparent places of the fixed stars. But for the heavenly bodies in our system, another correction called *parallax*, must be taken into account, to make their calculated places correspond with observation. A popular illustration of this subject may be obtained by considering the difference of situation in a body supposed to be viewed from the surface and from the centre of the earth. This last position alone is the real place of the body with respect to that of our planet (*e*).

(169.) Problem XVI. Given the right ascensions and declinations of two heavenly bodies, or their latitudes and longitudes, to find their distance asunder.

Example I. At apparent noon at Greenwich, Sept. 1st, 1831, the sun's longitude was 158° 15′ 16″, that of the moon 95° 25′ 43″, and the moon's latitude 3° 48′ 42″ south; required their true distance.

In the spherical triangle ABC, (fig. 34.) let A

represent the place of the sun, B that of the moon, and C the pole of the ecliptic; then we have given the side $a=93^\circ\ 48'\ 42''$, the side $b=90^\circ$, and the included angle C, the difference of longitude, $=62^\circ\ 49'\ 33''$, to determine the side c, the required distance.

For this purpose we have the equation,

$$\cos C = \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b};$$

but,

$$\cos b = 0,\ \sin b = 1;$$

$$\therefore \cos C = \frac{\cos c}{\sin a},$$

$$\text{or, } \cos c = \sin a \cdot \cos C.$$

$\sin 93^\circ\ 48'\ 42''$, log...... 9·9990382
$\cos 62^\circ\ 49'\ 33''$............ 9·6596280

$\cos c$......................... 9·6586662

$$\therefore c = 62^\circ\ 53'\ 27''\cdot 2.$$

Example II. Required the distance between the fixed stars Aldebaran, and α Lyræ, the right ascension of the former being $4^h\ 26^m\ 21^s$, and its declination

16° 10′ N., the right ascension of the latter being $18^h\ 31^m\ 17^s$, and its declination 38° 37′ 59″ N.

In the spherical triangle *A B C* (fig. 35.), in which *C* represents the pole of the equator, *A* the position of Aldebaran, and *B* that of *α* Lyræ; we have given the sides *a* and *b*, the north polar distances of the two stars, and the included angle $C = 148°\ 46'$; to find the opposite side *c*, the distance between the two stars.

To obtain this directly, either of the formulæ in (91.) or (92.) may be used, but as the solution by the latter will give rise to some observations which may be useful to the student, we shall employ it.

We have,

$$\tan\phi = \cos C \, . \tan b, \ \cos c = \frac{\cos b \, . \cos(a-\phi)}{\cos\phi},$$

which are thus computed,

cos 148° 46′, log......... 9 · 9319980
tan 51° 22′ 1″............ 10 · 0973257

tan φ...................... 10 · 0293237

The number corresponding to this logarithmic tangent, is 46° 55′ 58″; but since cos *C* is negative, we must consider tan *φ* negative, and therefore we may take for the value of *φ*, either —46° 55′ 58″, or

133° 4′ 2″ (9.) or (13.), the solution on either hypothesis being the same.

Adopting the latter value we have $(a-\varphi) = -59^\circ\, 14'\, 2''$; and we finish the computation as follows:

cos 59° 14′ 2″, log......	9·7088751
cos 51° 22′ 1″............	9·7954145
	19·5042896
cos 133° 4′ 2″............	9·8343291
cos c......................	9·6699605

Now since cos φ is negative, cos c is also negative; therefore the number answering to this logarithmic cosine must be the supplement of that in the tables; hence we have for the value of c

the arc 117° 53′ 5″.

By this problem the *lunar distances* in the Nautical Almanac are calculated. The right ascension and declination of the moon being obtained from the lunar tables, and those of the nine fixed stars, viz. α Arietis, Aldebaran, Pollux, Regulus, Spica ♍, Antares, α Aquilæ, Fomalhaut, and α Pegasi, whose situations are convenient to show the rapid motion of the moon, are determined by observation with more than ordinary care, the annual variations being continually applied. The moon's distance from the sun is obtained by the solution of a quarantal triangle; the data may be seen in Example I.

This is one of the most useful and celebrated problems in Astronomy. We shall give a few examples adapted to the year 1834.

LUNAR DISTANCES.

1. Aldebaran. Feb. 10th, 1834. Midnight[*].
✱'s A.R=66° 35′ 55″·1 ☽'s A.R=349° 56′ 21″
Dec.=16° 10′ 6″·2 N. Dec.=9° 37′ 34″·8 S.
True distance=80° 6′ 0″·5.

2. Regulus. Feb. 20th, 1834. Midnight.
✱'s A.R=149° 53′ 1″·6 ☽'s A.R=114° 16′ 40″·5
Dec.=12° 46′ 30″·1 N. Dec.=23° 10′ 12″·8 N.
Distance=35° 19′ 2″·5.

3. *α* Aquilæ. July 16th, 1834. Midnight.
✱'s A.R=295° 40′ 42″·5 ☽'s A.R=245° 7′ 56″·9
Dec.=8° 26′ 12″·4 N. Dec.=19° 41′ 9″·9 S.
Distance=57° 9′ 5″·7.

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[*] Apparent time at Greenwich.

4. α Arietis. Oct. 10th, 1834. Midnight.
✱'s A.R=29° 28′ 5″·2, ☽'s A.R=306° 8′ 59″·3
Dec.=22° 40′ 41″·3 N. Dec.=22° 53′ 37″·9 S.
Distance=92° 55′ 42″·6.

5. Pollux. Oct. 22nd, 1834. Midnight.
✱'s A.R=113° 47′ 39″·7 ☽'s A.R=87° 20′ 26″·2
Dec.=28° 25′ 9″·9 N. Dec.=24° 15′ 14″·7 N.
Distance=24° 1′ 14″·1.

6. Antares. Dec. 22nd, 1834. Midnight.
✱'s A.R=244° 48′ 47″·6 ☽'s A.R=176° 23′ 50″·1
Dec.=26° 3′ 21″ S. Dec.=7° 16′ 37″ N.
Distance=74° 12′ 21″·8.

SECTION XII.

ON THE LONGITUDE.

(170.) Longitude of a place upon the earth may be defined to be the *angle at the pole formed by the meridian of the place and the first meridian.* This angle is measured by the corresponding arc of the equator, and may be expressed either in degrees or in time. When the sun is on the meridian of any place to the west of Greenwich, or in west longitude, he has evidently passed the first meridian, and the time at Greenwich is afternoon: on the other hand, when it is apparent noon at a place in east longitude, the sun has not yet arrived at the first meridian. Hence this problem, which is of the utmost importance in Nautical Astronomy, consists in finding what o'clock it is at two places under different meridians at the same instant. The most obvious method, apparently, of effecting this, would be to determine the time at the place of the observer, which can readily be done by the sun's altitude (155.), and comparing it with a chronometer previously regulated to shew Greenwich time; the difference in time would thus be known, and consequently the longitude of the observer. But notwithstanding the plausibility of this plan, it is found to be liable, in practice, to errors which no precaution can altogether guard against. Chronometers must always be subject to variations from many obvious causes, and perpetually liable to accidents; so that an entire reliance upon them at sea would frequently be attended with fatal consequences. They are, however, of the utmost importance when constantly verified by some method, more regular in its operation, and less under the influence of accidents.

(171.) The method universally agreed upon as the most practical and certain, is by *lunar distances.* As the distance of the moon from the fixed stars which are situated near her path varies about half a degree every hour, her change of place is sufficiently great to be perceptible in a small portion of time. In the Nautical Almanac are given the distances of the moon from the sun, and from certain stars, for every third hour of Greenwich time; and, as her motion is sufficiently uniform for the purpose, the time for any intermediate distance is obtained by a simple proportion. Now if the distance of the moon from a certain star be observed at any place, with the corresponding time, a reference to the Nautical Almanac, which, in this case, stands in the stead of a perfect chronometer, will show the time at Greenwich when the same distance or event took place: hence the difference of time between the two places becomes known. But owing to the *observed* distance not being the *true* distance, the problem is extended to the solution of two spherical triangles.

(172.) Let *Z* (fig. 33.) be the zenith of the observer, *Z* ☽ the apparent zenith distance of the moon, *Z* ☉ that of the sun, and ☽ ☉ the apparent or observed distance between them. Now all the heavenly bodies are affected by refraction, which elevates them above their true places. And the bodies in our system are depressed below their true places by parallax.

The effects of refraction and parallax are greatest in the horizon, and vanish at the zenith. The parallax of the moon, owing to her proximity to the earth, is greater than the refraction; and hence the true altitude of this body will always be greater than the apparent altitude. On the other hand, the parallax of the sun is less than the refraction, and therefore the true altitude will be less than the apparent altitude. But as these effects operate only in a vertical direction, the angle at the zenith will remain unaltered.

Commencing now with the triangle Z ☽ ☉, we have given Z ☽, Z ☉, the apparent zenith distances, with ☽ ☉, the apparent distance between the two bodies, to find the angle ☽ Z ☉ : the true places m, S, being then found by applying the necessary corrections for refraction, parallax, &c.; in the triangle $Z m S$, we shall have given, the true zenith distances, $Z m$, $Z S$, with the angle Z, to find the side $m S$, the true lunar distance.

Example I. Suppose the apparent altitude of the moon to be 8° 26′ 13″, the true altitude 9° 20′ 45″; the apparent altitude of a star 35° 40′, its true altitude 35° 38′ 49″; also the apparent distance between them 31° 13′ 26″. Required the true distance.

True distance = 30° 23′ 56″.

Example II. Let the apparent zenith distance of the sun be 68° 25′, that of the moon 67° 45′, the corrected zenith distance of the sun 68° 27′ 16″, that of the moon 66° 53′ 28″, and the apparent distance 119° 20′ 34″. To find the true distance.

True distance = 118° 46′ 48″.

SECTION XIII.

ON THE PRINCIPLES OF DIALLING.

(173.) As the apparent diurnal motion of the sun about the axis of the world is at the rate of 15° in an hour, let us suppose *Z* (fig. 36.) to represent the zenith of a spectator, *H R* the horizon, *H Z P R* the meridian, and *P O S* the axis of the world; also let us imagine an hour circle to accompany the sun in his diurnal motion. Now when the sun is on the meridian, the shadow of the semi-axis *O P* will be cast or projected in the plane of this circle upon the line *O R*, the intersection of the planes of the meridian and horizon. At the hours one, two, three, &c., from the meridian, the shadow of *O P* will be cast in the planes of the respective hour circles upon the straight lines *O* 1, *O* 2, *O* 3, &c. These are called the one, two, three o'clock, &c., hour lines.

Now it is manifest, that when the sun is one hour, two hours, &c., distant from the meridian, the corresponding angles *R P* 1, *R P* 2, &c., will be, respectively, 15°, 30°, &c.

Hence, in the right angled spherical triangles $RP1$, $RP2$, &c., $PR=$ latitude of the place, $RP1=15^\circ$, $RP2=30^\circ$, &c., therefore,

$$\tan R1 = \sin \text{lat} . \tan 15^\circ$$

$$\tan R2 = \sin \text{lat} . \tan 30^\circ.$$

Similarly,

$$\tan R3 = \sin \text{lat} . \tan 45^\circ$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

In this manner we calculate the angles which the hour lines make with OR, the twelve o'clock hour line, or meridian.

(174.) Here we have made the axis of the world, the gnomon or style, and the rational horizon the plane of the dial; but if we take any other plane parallel to HR, the sensible horizon of the spectator, for instance, and draw a style parallel to OP, and which will therefore be inclined to the plane of the dial in an angle equal to the spectator's latitude; on account of the great distance of the sun we may conceive it to revolve about this style in the same way as about OP, and to cast a shadow upon the plane of this dial, in the same manner as upon the plane HR. This is called a *horizontal dial.*

(175.) Now, let ZN (fig. 37.) represent the plane of the prime vertical, and therefore the side next to H due south. Then for the same reason as before, if the angles $NS\,1$, $NS\,2$, &c. $=15°$, $30°$, &c., the shadow of the semi-axes, OS, will be cast upon the plane of the prime vertical in the lines $O\,1$, $O\,2$, &c., at one o'clock, two o'clock, &c.

Hence, in the right angled triangles $NS\,1$, $NS\,2$, &c., $NS=$ co-latitude of the place, $NS\,1=15°$, $NS\,2=30°$, &c., we have,

$$\tan N\,1 = \cos \text{lat} \,.\, \tan 15°$$
$$\tan N\,2 = \cos \text{lat} \,.\, \tan 30°.$$

Similarly,

$$\tan N\,3 = \cos \text{lat} \,.\, \tan 45°$$
$$\&\text{c.} \quad = \quad \&\text{c.}$$

In this way the angles which the hour lines make with the twelve o'clock hour line ON are calculated.

(176.) If, then, we take a plane which coincides with that of the prime vertical, and erect a style in the plane of the meridian, making an angle with the plane of the dial equal to the co-latitude of the place, we may, for the same reason as before, suppose the sun to revolve about this style, and to cast its shadow upon the plane of this dial, the hour angles of which are calculated as before.

This is a *vertical south dial.*

(177.) Generally, whatever be the inclination of the plane of the dial to that of the meridian, the hour angles may be calculated by (94).

(178.) We have hitherto considered the style to be a mathematical line; in order to compensate for its breadth, in the practical construction of dials, two lines must be drawn parallel to the meridian, one on each side; the distance between the lines, which are equally distant from the meridian, being equal to the thickness of the style. The hour angles are then laid off from these lines instead of the meridian.

(179.) To place a horizontal dial, find the time by the sun's altitude (155.), and set a well regulated watch to it; then when the watch shews twelve o'clock, the sun is upon the meridian; at that instant place the dial horizontally, to shew that time.

(180.) Another method consists in describing several concentric circles, and erecting at the centre, perpendicular to the plane, a gnomon, having a small, Nº 3.

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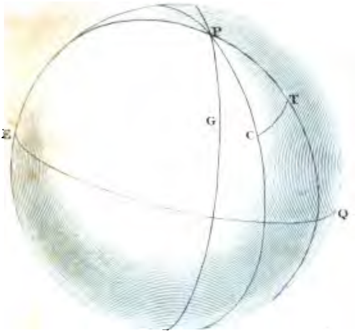

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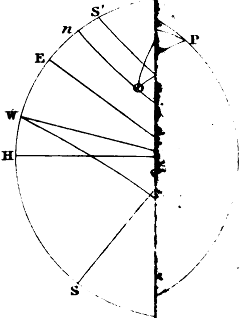

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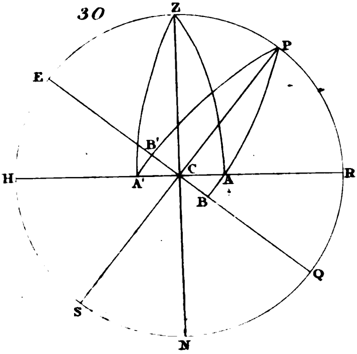

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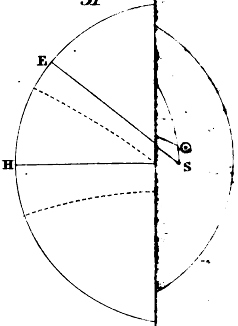

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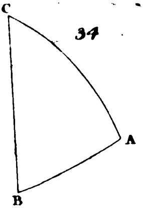

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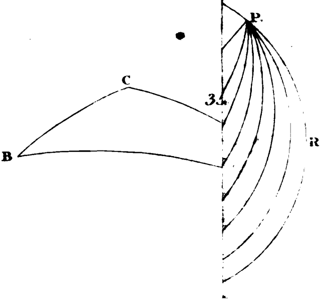

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round, well defined head. The shadow of the head will cross these circles, once in the forenoon and once in the afternoon; and, if the points be marked, and the corresponding arcs be bisected, the line joining the points of bisection will be the meridian line, with which the twelve o'clock hour line must coincide. It is, however, not susceptible of much accuracy except when the sun is near the solstices; for when the declination varies considerably, the sun is not at equal altitudes at equal times from the meridian.

SECTION XIV.

GEODESY, OR TRIGONOMETRICAL SURVEYING.

(181.) Trigonometrical surveys have been carried forward in different countries either for the purpose of mapping an extensive region, or of measuring the length of a degree in various latitudes, so as to determine the figure of the earth. They afford the best exemplifications of plane and spherical trigonometry; but as they likewise require peculiar formulæ, which would be unsuitable to the design of this work, we shall merely aim at describing generally the course of operations.

(182.) The first part is the measurement of a base, for which a plain, four or five miles in extent, is generally chosen; this line is measured with the most scrupulous accuracy. On Hounslow Heath a base was selected, and General Roy commenced the measurement with well seasoned deal rods, each twenty feet long, which were secured from bending in the most effectual manner; yet it was found that the expansion occasioned by moisture was such as to take away all confidence in the correctness of the final result. Afterwards glass tubes of the same length were substituted, and the temperature being always noticed, the proper corrections for expansion were applied. The base measured by these was found to be 27404·08 feet, or rather more than five miles. Several years afterwards, the same base was remeasured by General Mudge by means of a steel chain constructed by the celebrated artist Ramsden, the corrections for expansions by temperature being applied; the result of this measurement gave a difference of 2¾ inches from the former determination by means of the glass tubes; and, as it could not easily be determined which was nearer the truth, a mean between them was taken.

(183.) In the late French survey, the measuring rods consisted each of a rod of platina, and a rod of brass connected at one extremity, and lying upon each other, the expansions of the two metals being different, the difference of their expansions was observed, and the total expansion of one rod was found by a simple proportion.

(184.) Proper stations for signals being selected, the whole country is divided into triangles by lines conceived to connect the signals, and the angles of these triangles are observed, the first observation being made at the extremities of the measured base. In the late French and Swedish surveys, the angles were measured by the *repeating circle,* a very portable instrument, invented by Mayer, and greatly improved by Borda; but as the signals are seldom seen in the horizon, a calculation is necessary to obtain the horizontal angle. In England and in India, the horizontal angle was observed directly by a theodolite.

(185.) In all the principal triangles each of the three angles is observed; for the smaller triangles it is sufficient to observe two of its angles.

(186.) Beginning now with the measured base, we have the length of this line and the two angles at each extremity, to determine the distance of a signal from its extremities. Thus (in fig. 38.) we determine $A\,C$, $B\,C$: similarly $A\,D$, $B\,D$ are found, then $C\,D$ is determined. In the triangle $C\,E\,D$, we have similar data, and thus we continue the process through any number of triangles till we arrive at a plain upon which a base may be measured to verify the preceding observations and calculations. This is called a *base of verification.*

(187.) It is generally thought proper to choose the stations for signals such that the sides of the triangle may be greater than ten and less than twenty miles. In the English survey, however, the distance from Beachy Head to Dunnose, which formed one side of a triangle, was more than sixty-four miles. And in the extension of the late French survey to Spain, a triangle was formed to connect Spain with Iviza, one of the sides of which was nearly 100 miles.

(188.) The calculations are verified either by comparing the calculated length of a base with the measured length; or by comparing the distance between two signals, as computed from two webs of triangles commencing either from the same or different bases. Thus a base of verification was measured on Salisbury Plain of more than seven miles, and this differed only one inch from the result of the calculation carried on from the base on Hounslow Heath. Again, by a series of triangles commencing at Dunnose, in the Isle of Wight, and extending to Beacon Hill, near Clifton, in Yorkshire, a distance of nearly 200 miles, the error in a line twenty-two miles in length, does not exceed six feet.

(189.) For the purpose of mapping, the latitudes and longitudes of the principal stations should also be accurately determined; but those of the minor objects that have been observed may be obtained by a more expeditious method.

(190.) For the determination of a degree of latitude, the distance of two places under the same meridian must be accurately determined; and the latitude of each must be observed. The length of a degree corresponding to the middle point between the two places will be known by a simple proportion *. The actual magnitude of the earth, considered spherical, is determined in this manner. If we travel from south to north, till the pole is elevated or depressed one degree, we shall evidently have travelled over one degree of the earth's surface, which being measured and multiplied by 360 will give the number of miles in the earth's circumference, and hence the number of miles in the diameter of the earth will also be known. Connected with this object was the late French survey; their purpose being to determine the length of the terrestrial quadrant which passed through France, in order to found upon such an imperishable standard, a system

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* The length of a degree, in the direction of the meridian, in latitude 50° 10′, was found to be 365,058 feet.—*English Trigonometrical Survey*.

of weights and measures. The distance between the pole and equator being determined, the 10,000,000th part, or the metre[*], was made the standard of linear measure.

(191.) We shall terminate this subject with the following problem, which is useful in the art of taking plans, when the surface to be operated upon presents any sensible inequalities, and it is required to determine and delineate the principal positions with great accuracy.

Let *O M N* (fig. 40.) be three points in a plane inclined to the horizon; from these three points draw the perpendiculars *O D, M m, N n* to meet the horizontal plane in the points *D, m, n*. The objects *O, M, N,* will be represented on the horizontal plane by their projections *D, m, n*. Suppose now the angle *M O N,* and the inclinations of the two lines *O M, O N,* to the vertical *O D* were given, and that we had to find the angle of projection, *m D n*.

From the point *O* as a centre, with a radius$=1$, describe a spherical surface meeting the lines *O M, O N,* and the vertical *O D,* in the points *A, B, C,* we shall have a spherical triangle *A B C,* in which the three sides are known, to find the angle C, or the equal angle $m\,D\,n$.

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[*] This was found to be 39·383 inches.

Example. Suppose the angle $M\,O\,N = A\,B = 58^\circ$, the angle $D\,O\,M = A\,C = 88^\circ\ 18'$, and $D\,O\,N = B\,C = 94^\circ\ 53'$; we shall obtain the required angle by the formula

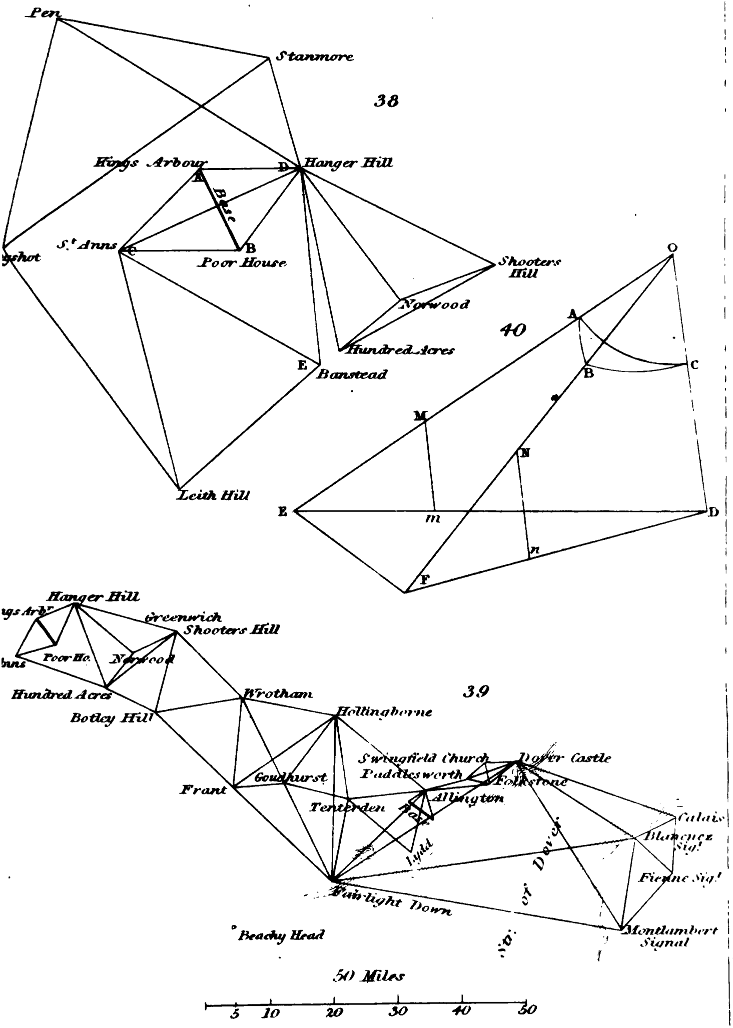
$$\sin\frac{C}{2} = \sqrt{\frac{\sin\overline{S-a}\,.\,\sin\overline{S-b}}{\sin a\,.\,\sin b}},$$

the calculation of which gives,

$$C = 57^\circ\ 40'\ 54''\cdot 8.$$

Hence the angle 58°, measured upon a plane inclined to the horizon, is reduced to 57° 40′ 54″·8, when it is projected upon the plane of the horizon.

Nº 4.



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NOTES.

NOTE (*a*), page 1.

THOUGH the centesimal division of the quadrant was immediately succeeded by corresponding trigonometrical tables of the most extensive scale, yet it was not attempted to be followed by any other nation: and even in France, it required all the authority of the government, and all the influence of Laplace, and other celebrated mathematicians by whom it was introduced, for many years to effect even its partial adoption. This might, however, have been expected from a project, notwithstanding its many and obvious advantages, that rendered useless all the trigonometrical and navigation tables, which, from having passed through so many hands, might be considered correct; that lessened, in a considerable degree, the value of the best instruments in observatories; and which at once sacrificed all the conveniences attending the continual trisection through the quadrant, so important to artists in the division of circular instruments. Indeed it would be difficult to imagine a more violent and extensive change than its universal adoption would have effected in all astronomical, mathematical, and geographical works. Its fate, however, may now be considered decided; the French mathematicians are gradually returning to the sexagesimal division, their power, gigantic as it was, having been found to be unequal to the task of destroying the prejudices, and altering the habits of a whole community.

To compare grades and degrees, we must observe that the circumference of the circle is divided into 400 and 360 equal parts respectively: hence a grade is to a degree as 9 is to 10. And the number of grades is to the number of degrees as 10 is to 9;

therefore, number of grades $=\frac{10}{9}$ or $\left(1+\frac{1}{9}\right)$ number of degrees,

and, number of degrees $=\frac{9}{10}$ or $\left(1-\frac{1}{10}\right)$ number of grades.

Example I. Convert $63^g\ 66197$ into degrees.

$$\begin{array}{rr} & 63\cdot 66197 \\ \text{one tenth} = & 6\cdot 366197 \\ \hline & 57\cdot 295773 \\ & = 57^\circ\ 17'\ 44''\cdot 7828 \end{array}$$

Example II. Reduce $57^\circ\ 17'\ 44''\cdot 7828$ into grades.

$$\begin{array}{rr} & 57^\circ\ 295773 \\ \text{one ninth} = & 6\ \ 366197 \\ \hline & 63^g\ 661970 \end{array}$$

Note (*b*), p. 73.

It seems not very easy to assign the reason why Napier, in framing his celebrated rules called the "Five Circular Parts," employed the complements of the hypothenuse and the complements of the angles, instead of the complements of the sides. Had he adopted the latter, and altered his rules to the following:—*product of radius and cosine of the middle part*

=*product of cotangents of the conjoined parts*, and=*product of sines of the disjoined parts;* they would have been more convenient in practice, since there would be fewer changes to make relating to the complements.

The science of Trigonometry is greatly indebted to this great man. In addition to his discovery of logarithms, by the aid of which the calculations which occupied months could be performed in as many days; and the final simplification of the decimal notation, he discovered the four equations in (90.) and (94.) which are commonly known by the name of Napier's Analogies. His discoveries also seem to be attended with the singular felicity of admitting of no improvement.

NOTE (*c*), pp. 33. 87.

It will be observed, in the solutions of plane and spherical triangles, that we have given severa formulæ for the same case. The reason of this is, that an angle cannot be accurately determined from its cosine when very small, nor from its sine when near 90°; but from its tangent it can generally be found with accuracy. Of the expressions, therefore, in (50.) and (51.), the first must not be employed when $\frac{C}{2}$ or C is very small; nor the second when $\frac{C}{2}$ is near 90°; but that in (52.) (except in extreme cases to be mentioned) may be always employed. The same observations apply when the expressions in (89.) are used.

In (47.) we have the expression $\cos A = \frac{b}{c}$ which is inaccurate if A is small, but we have,

$$1-\cos A=\frac{c-b}{c}\text{ and }1+\cos A=\frac{c+b}{c},$$

$$\text{therefore, }\frac{1-\cos A}{1+\cos A}\,(23.)\ \tan^2\frac{A}{2}=\frac{c-b}{c+b},$$

an expression which may be safely used.

In (48.), we have $\sin C=\frac{c}{a}\sin A$; if C is near 90° it is inaccurate.

Let $C=90^\circ\pm x$ then $\sin C$ or $\cos x=\frac{c}{a}\sin A$;

$$\text{therefore, as in the preceding }\tan^2\frac{x}{2}=\frac{1-\frac{c}{a}\sin A}{1+\frac{c}{a}\sin A}.$$

Now, $\frac{c}{a}$ in all cases of difficulty will be greater than 1 and less than $\frac{1}{\sin A}$: let $\frac{a}{c}=\sin\theta$,

$$\text{then, }\tan^2\frac{x}{2}=\frac{\sin\theta-\sin A}{\sin\theta+\sin A},\text{ or by (30.)}=\frac{\tan\frac{\theta-A}{2}}{\tan\frac{\theta+A}{2}};\qquad\times$$

which can be calculated with accuracy.

In (86.) equation 1, if a and b are very small, c cannot be accurately found from that formula; we must therefore take the equations 8 and 9, $\tan A=\frac{\tan a}{\sin b}$, and $\tan c=\frac{\tan b}{\cos A}$, by which c is found to the greatest accuracy.

In equation 9, $\cos A = \frac{\tan b}{\tan c}$; if A be small this is inaccurate;

but $1 - \cos A = \frac{\tan c - \tan b}{\tan c}$, and $1 + \cos A = \frac{\tan c + \tan b}{\tan c}$;

hence this expression must be used,

$$\frac{1-\cos A}{1+\cos A}, \text{ or } \tan^2 \frac{A}{2} = \frac{\tan c - \tan b}{\tan c + \tan b}$$

$$= \frac{\sin \overline{c-b}}{\sin \overline{c+b}} \quad (28.)$$

In (91.), (92.), (93.), when c is near 180°, these cases fail in accuracy: recurring, however, to the fundamental theorem,

$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$, we have,

$1 + \cos c = 1 + \cos a \,.\, \cos b + \sin a \,.\, \sin b - \sin a \,.\, \sin b$ $(1 - \cos C)$

$$= 1 + \cos(a+b) - \sin a \,.\, \sin b\,(1 - \cos C)$$

$$\therefore \cos^2 \frac{c}{2} = \cos^2 \frac{a+b}{2} - \sin a \,.\, \sin b \,.\, \sin^2 \frac{C}{2},$$

let $\sin a \,.\, \sin b \,.\, \sin^2 \frac{C}{2} = \sin^2 \theta$,

then $\cos^2 \frac{c}{2} = \cos^2 \frac{a-b}{2} - \sin^2 \theta =$

$$= \cos\left(\frac{a-b}{2} + \theta\right) \,.\, \cos\left(\frac{a-b}{2} - \theta\right) \quad (25.)$$

The ground of these cautions is, to avoid the imperfections of the common tables. If the tables are carried to seven decimals (the usual extent) the log sines cannot be trusted beyond 87° 17′; but if to ten decimals, they are correct as far as 89° 59′ 50″*. The general reason will be seen, by considering the series in (104.) expressing the values of the sine and cosine in terms of the arc. When the arc is very small, the value of the cosine converges very slowly; and when it is near 90°, the value of the sine converges equally slow; and since the logarithmic table of sines, cosines, &c., is only approximative, the difference in the numerical values of the cosine, or sine of arcs, near 0, or 90°, is less than can be appreciated by the tables. This inconvenience is commonly avoided, to a certain extent, by constructing a table for a few of the first degrees of the quadrant to every second, or to smaller intervals than the rest. It frequently happens, in making optical and other experiments, that we have to calculate or compare angles so minute, as to escape the reach altogether of the tables. In these cases the sine and tangent become confounded with the arc; and therefore the magnitude of the arc, in terms of radius 1, or the angle, is found by simple proportion.

Thus, in (47.) we have $\sin A = \frac{a}{c}$ and $\tan A = \frac{a}{b}$;

if a be very small, compared with b or c, $\sin A$ or $\tan A$ may be taken as the arc; and the magnitude of the arc, or angle, in seconds, will be obtained by dividing the numerical value by ·0000048481368; or subtracting from its logarithm 4·68557487.

NOTE (*d*), p. 130.

In the construction of trigonometrical tables, those processes, only, have been explained which depended upon the preceding part of the work; but it may be proper to inform the student, that no one would undertake their calculation without a moderate acquaintance with the Calculus of Finite Differences. And though the method of computing, by differences, all tables which are deduced from formulæ, or series, expressive of certain laws, is remarkably simple, and possesses peculiar advantages, yet the general principle upon which it is founded could not be understood by those for whom this treatise is intended, and therefore it has been omitted. But the student who has advanced somewhat beyond elementary subjects will find in Professor Airey's Treatise on Trigonometry *, a complete view of the best and most expeditious methods of forming trigonometrical tables. This excellent performance has been of great assistance in the compilation of various parts of the present work, and we cordially recommend it to the mathematical reader.

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* Airy's Trigonometry

The values of the trigonometrical lines in terms of the arc, being expressed by converging series, render it necessary to obtain the differences at shorter intervals in the commencement of the table, and at longer when the table becomse extensive. But this preparatory labour being effected, the rest of the computation is entirely mechanical. In the archives of the observatory of Paris, are preserved in manuscript, tables of logarithms and of logarithmic sines, tangents, &c. occupying seventeen folio volumes, which were computed by about seventy persons in a short period of time, entirely by addition and subtraction. "And it is remarkable, that nine tenths of these persons had no knowledge of arithmetic beyond its first two rules, which they were thus called upon to exercise; and that they were usually found to be more

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* Encyclopædia Metropolitana, Pure Sciences, Vol. I.

correct in their calculations than those who possessed a more extensive knowledge of the subject."*

But the grand and original idea of embodying in the construction of machinery, the practical operation of differences, enabling it to save the time, and thus direct the vigor of professional calculators to labour of a higher interest, was reserved for Mr. Babbage. A *calculating engine*, which by mere feeding with numbers at increasing intervals of time, brings out the finished calculations, and registers them by means of hardened steel dies upon a copper plate, without the possibility of error, is far advanced in its construction. This extraordinary machine, the triumph of science and mechanism, has had the labour of ten years bestowed upon it, and it is thought that probably two years more may elapse before it will be completed†.

NOTE (*e*), p. 168.

As the astronomical part of the work was only intended to be of an elementary nature, and principally designed to exemplify the use of Spherical Trigonometry, we must refer the reader for satisfactory explanations of parallax, refraction, and the other corrections affecting the altitudes of the heavenly bodies, to professed treatises on Astronomy. To be well acquainted, however, with this science, it will be necessary for the student to look farther than the mere elements of Mathematics as a preparatory step. He may, it is true, obtain a practical knowledge, that will enable him to work many problems, but this rather barren acquisition, when its novelty is worn off, will yield him little real satisfaction, and it will even be used with that timorousness which is inseparable from a mind which has not attained the truth. Without a knowledge of the higher analysis, however he may have been informed by the popular writers of the present day, he will not be enabled by any efforts of his own, or aid of others, to understand the reasons of the operations he may have occasion to employ, nor to take his stand in the ranks as a real observer of the beautiful simplicity of nature, and the perfect connexion which exists between the laws of motion and the great phenomena of the universe.

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* "Economy of Machinery and Manufactures," page 156.

† For an interesting account of this engine, and of the effects of a small model or trial machine, see Sir David Brewster's "Letters on Natural Magic," just published.

THE END.

G. Woodfall, Printer, Angel Court, Skinner Street, London.